FINITE TREE AUTOMATA AND ω-AUTOMATA

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ABSTRACT

Chapter I is a survey of finite automata as acceptors of finite labeled trees. Chapter II is a survey of finite automata as acceptors of infinite strings on a finite alphabet. Among the automata models considered in Chapter II are those used by McNaughton, Buchi, and Landweber. In Chapter II we also consider several new automata models based on a notion of a run of a finite automaton on an infinite string suggested by Professor A.R. Meyer in private communication. We show that these new models are all equivalent to various previously formulated models.

M.O. Rabin has published two solutions of the emptiness problem for finite automata operating on infinite labeled trees. Appendices I and II contain a new solution of this emptiness problem. This new solution was obtained jointly by the author and Charles Rackoff.

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Introduction

In 1969 M.O. Rabin [8] used finite automata on infinite trees to give a decision procedure for the monadic second-order theory of two successor functions. This is a powerful result and has as corollaries decision procedures for several other interesting theories whose decision problems were previously open. It, also, solved several other open problems. Rabin's work has attracted considerable attention from mathematicians who are otherwise not very interested in the notion of a finite automaton.

We believe that the proper way to begin one's study of finite automata on infinite trees and of Rabin's work is by studying finite automata on finite trees and finite automata on infinite sequences. It is hoped that this thesis will aid the reader in these preliminary studies.

The most interesting new result contained in this thesis was obtained jointly by the author and Charles Rackoff and is presented in Appendices I and II. We reduce the emptiness problem for finite automata on infinite trees (either as defined by Rabin in [8] using the designated subset acceptance condition, or as defined by Rabin in [7] using the Ω acceptance condition) to the emptiness problem for finite automata on finite trees.

Every proof in this paper is effective in the sense that:

1) When the existence of a finite automaton with certain properties is asserted given the existence of another (other) finite automaton (s), then the proof consists in determining the finite automaton with the asserted properties from the given finite automaton (s).

2) When the existence of a characterization of a set is asserted given another characterization of it, the proof consists in determining the new characterization from the old characterization.

The principal results in Chapter I are summarized by the following:

A 1-f.a.f.t. is a leaf-up nondeterministic finite automaton on finite Σ-trees.

A 2-f.a.f.t. is a leaf-up deterministic finite automaton for finite Σ-trees.

A 3-f.a.f.t. is a root-down nondeterministic finite automaton on finite Σ-trees.

A 4-f.a.f.t. is a root-down deterministic finite automaton on finite Σ-trees.

1-f.a.f.t. $\equiv$ 2-f.a.f.t. $\equiv$ 3-f.a.f.t. $\supset$ 4-f.a.f.t.

	3-f.a.f.t.	4-f.a.f.t.
union	closed	no
intersection	closed	closed
complementation	closed	no
projection	closed	no
cylindrification	closed	closed

TABLE 1

The principal results of Chapter II are summarized by the following. Those unfamiliar with the usual definition of a finite automaton run on an infinite sequence should refer to Chapter II. We abbreviate nondeterministic finite automata as n.f.a., and deterministic finite automaton as d.f.a.

For $i \in \{1, 1', 2, 2', 3, 4\}$, a run r on an infinite sequence of an i-n.f.a. (i-d.f.a.) $\mathfrak{M}$ with state set S is an accepting run if r is i-accepting, where

r is 1-accepting with respect to $F \subseteq S$ if

$(\exists t) \;\; r(t) \in F$,

r is 1'-accepting with respect to $F \subseteq S$ if

$(\forall t) \;\; r(t) \in F$,

r is 2-accepting with respect to $F \subseteq S$ if

$In(r) \cap F \neq \emptyset$,

r is 2'-accepting with respect to $\mathcal{F} \subseteq P(S)$ if

$(\exists F \in \mathcal{F}) \;\; In(r) \subseteq F$,

r is 3-accepting with respect to $\mathcal{F} \subseteq P(S)$ if

$In(r) \in \mathcal{F}$,

r is 4-accepting with respect to $\Omega = ((R_i, G_i))_{i<n}$, if for some $i < n$,

$In(r) \cap R_i = \emptyset$ and $In(r) \cap G_i \neq \emptyset$.

1-n.f.a. ≡ 1-d.f.a.

1'-n.f.a. ≡ 1'-d.f.a.

2'-n.f.a. ≡ 2'-d.f.a.

2-n.f.a. ≡ 3-n.f.a. ≡ 3-d.f.a. ≡ 4-n.f.a. ≡ 4-d.f.a.

	1'-f.a.	2-d.f.a.	2'-f.a.	3-f.a.
1-f.a.	incomp.	1 $\subset$ 2-d.	1 $\subset$ 2'	1 $\subset$ 3
1'-f.a.		1' $\subset$ 2-d.	1' $\subset$ 2'	1' $\subset$ 3
2-d.f.a.			incomp.	2d. $\subset$ 3
2'-f.a.				2' $\subset$ 3

	1-f.a.	1'-f.a.	2-d.f.a.	2'-f.a.	3-f.a.
union	yes	yes	yes	yes	yes
intersection	yes	yes	yes	yes	yes
complementation	no	no	no	no	yes
projection	yes	yes	no	yes	yes
cylindrification	yes	yes	yes	yes	yes

TABLE 2

CHAPTER I

Finite Automata on Finite Trees

SECTION I INTRODUCTION

In 1965 finite automata on finite trees were first used by J.E. Doner [2] who first applied them to obtain a decision procedure for the weak monadic second-order theory of two successor functions. Thatcher and Wright [11, 12] independently developed finite automata on finite trees and noticed the same application.

Finite automata on finite trees are no harder to visualize and understand than finite automata on finite sequences, and, in fact, the various finite automata models on finite trees have just those properties which one familiar with finite automata on finite sequences would expect them to have.

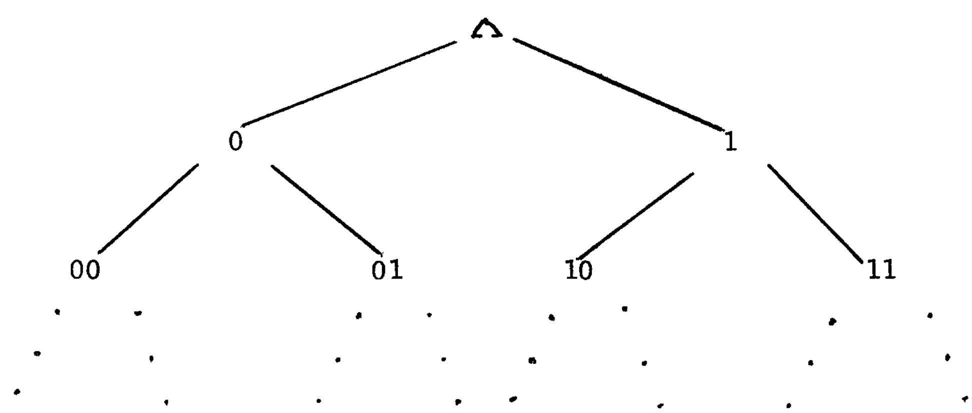
SECTION II DEFINITIONS

We will use the usual set theoretic notation throughout this paper. A function $f\colon A \to B$ is a subset $f \subseteq A \times B$ such that 1) for all $a \in A$ there is $(a,b) \in f$, for some $b \in B$, 2) for all $a \in A$, $b, c \in B$, $(a,b) \in f$ and $(a,c) \in f$ implies $b = c$. Sometimes we will describe a mapping by the notation $x \mapsto f(x)$, $x \in A$, which indicates that $x \in A$, x is mapped into $f(x)$. If $f\colon A \to B$ then A and $f(A) = \{f(a) \mid a \in A\}$ are called, respectively, the <u>domain</u> and the <u>range</u> of f. If $f\colon A \to B$ and $C \subseteq A$, then $f|C$ will denote the restriction $f \cap (C \times B)$ of f to C.

The cardinality of a set A will be denoted by c(A). The set of all subsets of a set A will be denoted by P(A). For α an ordinal we let the set $[\alpha] = \{\beta \mid \beta < \alpha\}$ of all smaller ordinals. We will use $\mathscr{N}$ to denote $[\omega] = \{0,1,2, \ldots\}$.

For the set A and n an integer, A^n is the set of all n-termed sequences of elements of A. That is, $A^n = \{\psi \mid \psi: \{1,2,\ldots, n\} \to A\}$. Let A be a set, n an integer, and $1 \le i \le n$. The <u>projection onto the ith coordinate</u> is the mapping $p_i: A^n \to A$ such that $p_i((x_1, \ldots, x_n)) = x_i$. Strictly speaking, projections such as $(x,y) \mapsto y$ and $(x,y,z) \mapsto y$ are different mappings, but we will denote both by p_2.

The <u>infinite binary tree</u> is the set $T = \{0,1\}^*$ of all finite strings of zeros and ones. The elements $x \in T$ are the <u>nodes</u> of T. For $x \in T$, the nodes x0, x1 are called the <u>immediate successors of x</u>. The empty word is called the <u>root</u> of T. Our language suggests the following picture. The highest node of T is the root Λ. The root branches down to the right into the node 0 and to the left into the node 1. The node 0 branches into 00 and 01; the node 1 branches into 10 and 11; and so on ad infinitum.



Definition: On T we define a partial ordering by $x \leq y$ iff there exists a z such that $y = xz$. If $x \leq y$ and $x \neq y$ then we shall write $x < y$.

Definition: For $x \in T$, the subtree T_x with root x is defined by $T_x = \{y \mid y \in T, x \leq y\}$. Note that $T_\Lambda = T$.

Definition: A path π of a tree T_x is a set $\pi \subset T_x$ satisfying 1) $x \in \pi$, 2) for $y \in \pi$, either $y0 \in \pi$ or $y1 \in \pi$, but not both, 3) π is a minimal subset of T_x satisfying 1) and 2).

Definition: A subset $F \subset T_x$ is called a frontier of T_x if for every path $\pi \subset T_x$ we have $c(\pi \cap F) = 1$.

It is easily seen that if $F \subset T_x$ is a frontier, then F is finite.

Definition: A finite frontiered tree with root z is a set $E_z = \{x \mid z \leq x$ & $x \leq y$, for some $y \in F\}$ where F is a fixed frontier of T_z. F is called the frontier of E_z and is denoted $Ft(E_z)$. By "finite tree" we will mean a finite frontiered tree. When the root is Λ we will often write E rather than E_Λ.

Definition: A finite Σ(labeled)-tree is a pair (v, E_z) where $E_z \subset T_z$ is a finite frontiered tree with root z and $v: E_z - Ft(E_z) \to \Sigma$.

Definition: The set of all finite Σ-trees with root Λ will be denoted Y_Σ.

Definition: The projection $p_1(A)$ of a set $A \subseteq Y_{\Sigma_1 \times \Sigma_2}$, is $p_1(A) = \{(p_1 v, E) \mid (v,E) \in A\} \subseteq Y_{\Sigma_1}$. The Σ_1-cylindrification of a set $B \subseteq Y_{\Sigma_1}$ is the largest set $A \subseteq Y_{\Sigma_1 \times \Sigma_2}$ such that $p_1(A) = B$.

The complement A^c of a set $A \subseteq Y_\Sigma$ is $A^c = Y_\Sigma - A$.

If (v, E_x) is a finite Σ-tree and $y \in E_x$, then the induced subtree $(v \mid E_x \cap T_y, E_x \cap T_y)$ will be denoted $(v, E_x \cap T_y)$.

Definition: An n-J(joining)-table on finite Σ-trees is a system $\mathfrak{A}' = \langle S, \Sigma, M, s_0 \rangle$, where S is the finite state set, Σ is the finite label set, $M: S \times S \times \Sigma \to P(S)$ is the state transition function, and $s_0 \in S$ is the initial state.

A d-J-table is an n-J-table with $M: S \times S \times \Sigma \to \{\{s\} \mid s \in S\}$.

An $\mathfrak{A}'$-run on Σ-tree $e = (v, E_x)$ is any mapping $r: E_x \to S$ such that 1) $r(Ft(E_x)) = \{s_0\}$, and 2) for all $y \in E_x - Ft(E_x)$, $r(y) \in M(r(y0), r(y1), v(y))$.

Definition: An n-S(splitting)-table on finite Σ-trees is a system $\mathfrak{A}' = \langle S, \Sigma, M, s_0 \rangle$, where S is the finite state set, Σ is the finite label set, $M: S \times \Sigma \to P(S \times S)$ is the state transition function, and $s_0 \in S$ is the initial state.

A d-S-table is an n-S-table with $M: S \times \Sigma \to \{\{(s_1, s_2)\} \mid (s_1, s_2) \in S \times S\}$.

An $\mathfrak{A}'$-run on Σ-tree $e = (v, E_x)$ is any mapping $r: E_x \to S$ such that 1) $r(x) = s_0$, and 2) for all $y \in E_x - Ft(E_x)$, $(r(y0), r(y1)) \in M(r(y), v(y))$.

We also talk about an $\mathfrak{A}$-run of an f.a.f.t. $\mathfrak{A}$ on a finite labeled tree meaning an $\mathfrak{A}'$-run of the associated J(S)-table $\mathfrak{A}'$. The set of all $\mathfrak{A}$-runs on e is denoted $Rn(\mathfrak{A}, e)$.

Definition: Leaf-up nondeterministic finite automaton on finite Σ-trees:

A 1-f.a.f.t. on finite Σ-trees is a system $\mathfrak{A} = \langle S, \Sigma, M, s_0, F \rangle$, where $\langle S, \Sigma, M, s_0 \rangle$ is an n-J-table, and $F \subseteq S$ is the set of designated states.

1-f.a.f.t. $\mathfrak{A}$ accepts finite Σ-tree $e = (v, E_x)$ if there exists an $\mathfrak{A}$-run r on e such that $r(x) \in F$.

Definition: Leaf-up deterministic finite automaton on finite Σ-trees:

A 2-f.a.f.t. on finite Σ-trees is a system $\mathfrak{A} = \langle S, \Sigma, M, s_0, F \rangle$, where $\langle S, \Sigma, M, s_0 \rangle$ is a d-J-table, and $F \subseteq S$ is the set of designated states.

2-f.a.f.t. $\mathfrak{A}$ accepts finite Σ-tree $e = (v, E_x)$ if there exists an $\mathfrak{A}$-run r on e such that $r(x) \in F$.

Definition: Root-down nondeterministic finite automaton on finite Σ-trees:

A 3-f.a.f.t. on finite Σ-trees is a system $\mathfrak{A} = \langle S, \Sigma, M, s_0, F \rangle$, where $\langle S, \Sigma, M, s_0 \rangle$ is an n-S-table, and $F \subseteq S$ is the set of designated states.

3-f.a.f.t. $\mathfrak{A}$ accepts finite Σ-tree $e = (v, E_x)$ if there exists an $\mathfrak{A}$-run r on e such that $r(Ft(E_x)) \subseteq F$.

Definition: Root-down deterministic finite automaton on finite Σ-trees:

A 4-f.a.f.t. on finite Σ-trees is a system $\mathfrak{A} = \langle S, \Sigma, M, s_0, F \rangle$, where $\langle S, \Sigma, M, s_0 \rangle$ is a d-S-table, and $F \subseteq S$ is the set of designated states.

4-f.a.f.t. $\mathfrak{A}$ accepts finite Σ-tree $e = (v,E_x)$ if there exists an $\mathfrak{A}$-run r on e such that $r(Ft(E_x)) \subseteq F$.

Definition: The set $T(\mathfrak{A})$ of finite Σ-trees defined by $\mathfrak{A}$ is $T(\mathfrak{A}) = \{(v,E) \mid (v,E) \text{ is accepted by } \mathfrak{A}\}$.

Definition: A set $A \subseteq Y_\Sigma$ is i-f.a.f.t. definable if there is an i-f.a.f.t. $\mathfrak{A}$ such that $T(\mathfrak{A}) = A$.

Note that the above definitions are not as general as you would expect. Though the notion of finite tree was defined so that a finite tree may have any root $x \in T$, the above definitions of $T(\mathfrak{A})$ and Y_Σ are in terms of finite trees with root Λ. We are forced to consider finite trees not rooted at Λ by later proofs in which we find it convenient to look at finite subtrees of a finite tree. However, we follow Rabin [9, 10] and Thatcher and Wright [12] (who define finite trees so that all of their finite trees have root Λ) by restricting our definitions of $T(\mathfrak{A})$ and Y_Σ. This eliminates several awkward notational problems.

Definition: We say i-f.a.f.t. $\mathfrak{A}_1$ is equivalent to j-f.a.f.t. $\mathfrak{A}_2$ if $T(\mathfrak{A}_1) = T(\mathfrak{A}_2)$.

Definition: We will say i-f.a.f.t. are closed under union (intersection, complementation, projection, cylindrification) if for all i-f.a.f.t. on Σ-trees $\mathfrak{A}_1$, $\mathfrak{A}_2$, there exists an i-f.a.f.t. $\mathfrak{A}_3$ such that $T(\mathfrak{A}_3) = T(\mathfrak{A}_1) \cup T(\mathfrak{A}_2)$

(intersection: $T(\mathfrak{A}_3) = T(\mathfrak{A}_1) \cap T(\mathfrak{A}_2)$,

complementation: $T(\mathfrak{A}_3) = Y_\Sigma - T(\mathfrak{A}_1)$,

projection: $T(\mathfrak{A}_3) = p_1(T(\mathfrak{A}_1))$, where $\Sigma = \Sigma_1 \times \Sigma_2$,

cylindrification: $T(\mathfrak{A}_3) = \Sigma_2$-cylindrification of $T(\mathfrak{A}_1)$, where $\Sigma = \Sigma_1$,

respectively).

Definition: We will say i-f.a.f.t. are equivalent (in defining power) to j-f.a.f.t. (denoted i-f.a.f.t. $\equiv$ j-f.a.f.t.) if the family of i-f.a.f.t. definable sets is the family of j-f.a.f.t. definable sets.

We will say i-f.a.f.t. are weaker or equivalent to j-f.a.f.t. (denoted i-f.a.f.t. $\subseteq$ j-f.a.f.t.) if the family of i-f.a.f.t. definable sets is a subset of the family of j-f.a.f.t. definable sets.

We will say i-f.a.f.t. are strictly weaker than j-f.a.f.t. (denoted i-f.a.f.t. $\subset$ j-f.a.f.t.) if the family of i-f.a.f.t. definable sets is a proper subset of the family of j-f.a.f.t. definable sets.

We will say i-f.a.f.t. and j-f.a.f.t. are incomparable if the family of i-f.a.f.t. definable sets is not a subset of the family of j-f.a.f.t. definable sets, and vice versa.

SECTION III FINITE AUTOMATA ON FINITE TREES

In [12] Thatcher and Wright use both 1-f.a.f.t. and 2-f.a.f.t. . In [10] Rabin uses a finite automata on finite trees model which is equivalent to 3-f.a.f.t. (In fact, Rabin uses 3-f.a.f.t. restricted as indicated in Theorem 3 of this section.) In [9] Rabin defines 2-f.a.f.t.

The following theorems establish the closure properties of all our models and their relative powers. We do not state the immediate corollaries of each theorem. Instead we summarize in Figure 1 all of our theorems and their immediate corollaries.

Theorem 1: 1-f.a.f.t. ≡ 3-f.a.f.t.

Proof: Given 3-f.a.f.t. $\mathfrak{A} = < S, \Sigma, M, s_0, F >$. Define 3-f.a.f.t. $\mathfrak{A}_1 = < S \cup \{f_1\}, \Sigma, M_1, s_0, \{f_1\} >$, where for all $s_1, s_2, s_3 \in S$, and all $\sigma \in \Sigma$, $(s_2, s_3) \in M_1(s_1, \sigma)$ if $(s_2, s_3) \in M(s_1, \sigma)$,

$$(f_1, s_3) \in M_1(s_1, \sigma) \text{ if } M(s_1, \sigma) \cap (F \times \{s_3\}) \neq \emptyset,$$
$$(s_2, f_1) \in M_1(s_1, \sigma) \text{ if } M(s_1, \sigma) \cap (\{s_2\} \times F) \neq \emptyset,$$
$$(f_1, f_1) \in M_1(s_1, \sigma) \text{ if } M(s_1, \sigma) \cap (F \times F) \neq \emptyset,$$

and for all $\sigma \in \Sigma$, $M_1(f_1, \sigma) = \emptyset$. Clearly, $T(\mathfrak{A}_1) \supseteq T(\mathfrak{A})$. For all finite Σ-trees $e = (v,E)$ and all accepting $\mathfrak{A}_1$-runs r_1 on e, we have for all $x \in E\text{-}Ft(E)$, $r_1(x) \in S$ and $r(Ft(E)) = \{f_1\}$. Hence, by the definition of M_1, there exists an $\mathfrak{A}$-run r on e such that for all $x \in E\text{-}Ft(E)$, $r(x) = r_1(x)$, and $r(Ft(E)) \subseteq F$. Therefore, $T(\mathfrak{A}_1) \subseteq T(\mathfrak{A})$, and hence, $T(\mathfrak{A}_1) = T(\mathfrak{A})$.

Define 1-f.a.f.t. $\mathfrak{A}_2 = < S \cup \{f_1\}, \Sigma, M_2, f_1, \{s_0\} >$, where for all $s_1, s_2, s_3 \in S \cup \{f_1\}$, and all $\sigma \in \Sigma$, $s_3 \in M_2(s_1, s_2, \sigma)$ if $(s_1, s_2) \in M_1(s_3, \sigma)$. Clearly, $T(\mathfrak{A}_2) = T(\mathfrak{A}_1)$, because for every finite Σ-tree e, every $\mathfrak{A}_1$-run on e is an $\mathfrak{A}_2$-run on e and vice versa, and a run is an accepting $\mathfrak{A}_1$-run iff it is an accepting $\mathfrak{A}_2$-run.

Hence, 3-f.a.f.t. ⊆ 1-f.a.f.t.

Given 1-f.a.f.t. $\mathfrak{A}_3 = < S_3, \Sigma, M_3, s_{30}, F_3 >$. Define 1-f.a.f.t. $\mathfrak{A}_4 = < S_3 \cup \{f_4\}, \Sigma, M_4, s_{30}, \{f_4\} >$, where for all $s_1, s_2, s_3 \in S_3$, and all $\sigma \in \Sigma$, $s_3 \in M_4(s_1, s_2, \sigma)$ if $s_3 \in M_3(s_1, s_2, \sigma)$,

$$f_4 \in M_4(s_1, s_2, \sigma) \text{ if } M_3(s_1, s_2, \sigma) \cap F_3 \neq \phi,$$

and for all $s_1 \in S_3$, and all $\sigma \in \Sigma$, $M_4(f_4, s_1, \sigma) = M_4(s_1, f_4, \sigma) = \phi$. Clearly, $T(\mathfrak{A}_4) \supseteq T(\mathfrak{A}_3)$. For all finite Σ-trees $e = (v,E)$ and all accepting $\mathfrak{A}_4$-runs r_4 on e, we have $r_4(\Lambda) = f_4$ and $r_4(E- \{\Lambda\}) \subseteq S_3$. Hence, from the definition of M_4, if there is an accepting $\mathfrak{A}_4$-run on e, then there exists an $\mathfrak{A}_3$-run r_3 on e such that $r_3(\Lambda) \in F_3$. That is, an accepting $\mathfrak{A}_3$-run on e. Hence, $T(\mathfrak{A}_3) \supseteq T(\mathfrak{A}_4)$, and therefore, $T(\mathfrak{A}_4) = T(\mathfrak{A}_3)$.

Define 3-f.a.f.t. $\mathfrak{A}_5 = < S_3 \cup \{f_4\}, \Sigma, M_5, f_4, \{s_{30}\} >$, where for all $s_1, s_2, s_3 \in S_3 \cup \{f_4\}$, and all $\sigma \in \Sigma$, $(s_2, s_3) \in M_5(s_1, \sigma)$ if $s_1 \in M_4(s_2, s_3, \sigma)$. Clearly, $T(\mathfrak{A}_4) = T(\mathfrak{A}_5)$, because for every finite Σ-tree e, every $\mathfrak{A}_4$-run on e is an $\mathfrak{A}_5$-run on e and vice versa, and a run is an accepting $\mathfrak{A}_4$-run iff it is an accepting $\mathfrak{A}_5$-run.

Hence, 3-f.a.f.t. $\supseteq$ 1-f.a.f.t. □

The constructions of $\mathfrak{A}_1$ and $\mathfrak{A}_4$ in the preceding proof immediatly give the following theorems which we state without further proof.

<u>Theorem 2</u>: Given any 1-f.a.f.t. $\mathfrak{A}$ on finite Σ-trees, we can determine an equivalent 1-f.a.f.t. $\mathfrak{A}_1 = < S_1, \Sigma, M_1, s_{10}, \{f_1\} >$, where M_1: $(S_1-\{f_1\}) \times (S_1-\{f_1\}) \times \Sigma \to P(S_1)$.

Theorem 3: Given any 3-f.a.f.t. $\mathfrak{A}$ on finite Σ-trees, we can determine an equivalent 3-f.a.f.t. $\mathfrak{A}_1 = < S_1, \Sigma, M_1, s_{10}, \{f_1\} >$, where M_1: $(S_1 - \{f_1\}) \times \Sigma \rightarrow P(S_1 \times S_1)$.

It is easily shown that Theorem 2 does not hold for 2-f.a.f.t.'s, and Theorem 3 does not hold for 4-f.a.f.t.'s by showing that $B_1 = \{(v,E) \in Y_{\{0,1\}} \mid v(E-Ft(E)) = 1 \text{ or } v(E-Ft(E)) = 0\}$ is both 2-f.a.f.t. and 4-f.a.f.t. definable, but that no 2-f.a.f.t. nor 4-f.a.f.t. with only one designated state defines B_1.

Theorem 4: 1-f.a.f.t. $\equiv$ 2-f.a.f.t.

Proof: Every 2-f.a.f.t. is a 1-f.a.f.t. . Hence, we have immediately 2-f.a.f.t. $\subseteq$ 1-f.a.f.t.

Given 1-f.a.f.t. $\mathfrak{A} = < S, \Sigma, M, s_0, F >$. Define 2-f.a.f.t. $\mathfrak{A}_1 = < P(S), \Sigma, M_1, \{s_0\}, F_1 >$, where for all $\Delta_1, \Delta_2 \subseteq S$, and all $\sigma \in \Sigma$, $M_1(\Delta_1, \Delta_2, \sigma) = \{\{s_3 \mid (\exists s_1 \in \Delta_1)(\exists s_2 \in \Delta_2)(s_3 \in M(s_1, s_2, \sigma))\}\}$, and $F_1 = \{\Delta \subseteq S \mid \Delta \cap F \neq \emptyset\}$.

We prove by induction that for every finite Σ-tree $e = (v,E)$, the $\mathfrak{A}_1$-run r_1 on e is such that

$$\text{(I)} \quad (\forall x \in E) \quad r_1(x) = \bigcup_{r \in Rn(\mathfrak{A}, e)} \{r(x)\} .$$

Basis of the induction: For all $r \in Rn(\mathfrak{A}, e)$, and all $x \in Ft(E)$, we have $r(x) = s_0$. We also have for all $x \in Ft(E)$, $r_1(x) = \{s_0\}$.

Induction hypothesis: For some $x \in E\text{-}Ft(E)$, $r_1(x0) = \bigcup_{r \in Rn(\mathfrak{A},e)} \{r(x0)\}$ and $r_1(x1) = \bigcup_{r \in Rn(\mathfrak{A},e)} \{r(x1)\}$.

Induction step: By the definition of $\mathfrak{A}$-run for all $r \in Rn(\mathfrak{A},e)$, we have $r(x) \in M(r(x0), r(x1), v(x))$. By the definition of M_1 we then have $r_1(x) = M_1(r_1(x0), r_1(x1), v(x)) = \{s \mid (\exists r \in Rn(\mathfrak{A}, e))(s \in M(r(x0), r(x1), v(x))\} = \bigcup_{r \in Rn(\mathfrak{A},e)} \{r(x)\}$. This completes the induction.

Suppose $e \in T(\mathfrak{A}_1)$. Then the $\mathfrak{A}_1$-run r_1 on e is accepting, i.e., $r_1(\Lambda) \in F_1$. By the definition of F_1 and by (I) there is an $\mathfrak{A}$-run r on e such that $r(\Lambda) \in F$. That is, r is an accepting $\mathfrak{A}$-run, and $e \in T(\mathfrak{A})$.

Suppose $e \in T(\mathfrak{A})$. Then there is an accepting $\mathfrak{A}$-run r on e, i.e., $r(\Lambda) \in F$. By (I) $r(\Lambda) \in r_1(\Lambda)$, where r_1 is the $\mathfrak{A}_1$-run on e. Hence, by the definition of F_1, $r_1(\Lambda) \in F_1$, and hence, r_1 is an accepting $\mathfrak{A}_1$-run on e. Therefore, $e \in T(\mathfrak{A}_1)$. □

Theorem 5: 3-f.a.f.t. are closed under union and projection.

Proof: The constructions will be indicated. The reader may easily complete the proofs.

Given 3-f.a.f.t.'s $\mathfrak{A}_1 = < S_1, \Sigma, M_1, s_{10}, F_1 >$ and $\mathfrak{A}_2 = < S_2, \Sigma, M_2, s_{20}, F_2 >$.

Define 3-f.a.f.t. $\mathfrak{A} = < S_1 \cup S_2 \cup \{s_0\}, \Sigma, M, s_0, F_1 \cup F_2 >$, where for all $s_1, s_2 \in S_1 \cup S_2$, and all $\sigma \in \Sigma$, $(s_1, s_2) \in M(s_0, \sigma)$ if $(s_1, s_2) \in M_1(s_{10}, \sigma)$ or $(s_1, s_2) \in M_2(s_{20}, \sigma)$, for all $s_1, s_2, s_3 \in S_1$, and all $\sigma \in \Sigma$, $(s_1, s_2) \in M(s_3, \sigma)$ if $(s_1, s_2) \in M_1(s_3, \sigma)$, and for all $s_1, s_2, s_3 \in S_2$, and all $\sigma \in \Sigma$, $(s_1, s_2) \in M(s_3, \sigma)$ if $(s_1, s_2) \in M_2(s_3, \sigma)$. $T(\mathfrak{A}) = T(\mathfrak{A}_1) \cup T(\mathfrak{A}_2)$.

Given 3-f.a.f.t. $\mathfrak{A}_3 = < S_3, \Sigma^2, M_3, s_{30}, F_3 >$. Define 3-f.a.f.t. $\mathfrak{A}_4 = < S_3, \Sigma, M_4, s_{30}, F_3 >$, where for all $s \in S_3$, and all $\sigma \in \Sigma$, $M_4(s, \sigma) = \bigcup_{\sigma_1 \in \Sigma} M_3(s, (\sigma_1, \sigma))$. $T(\mathfrak{A}_4) = p_2 T(\mathfrak{A}_3)$.

1-f.a.f.t. may be shown to be closed under union and projection by the obvious constructions corresponding to those in the preceding proof. Since we have 1-f.a.f.t. closed under union and projection as an immediate corollary of Theorems 1 and 5, it is unnecessary that we do these constructions.

Theorem 6: 3-f.a.f.t. and 4-f.a.f.t. are closed under intersection.

Proof: The construction will be indicated. The reader may easily complete the proof.

Given 3-f.a.f.t.'s (4-f.a.f.t.'s) $\mathfrak{A}_1 = < S_1, \Sigma, M_1, s_{10}, F_1 >$ and $\mathfrak{A}_2 = < S_2, \Sigma, M_2, s_{20}, F_2 >$.

Define 3-f.a.f.t. (4-f.a.f.t., respectively) $\mathfrak{A}_5 = < S_1 \times S_2, \Sigma, M_5, (s_{10}, s_{20}), F_1 \times F_2 >$, where for all $s_1, s_3, s_5 \in S_1$, all $s_2, s_4, s_6 \in S_2$, and all $\sigma \in \Sigma$, $((s_3, s_4), (s_5, s_6)) \in M_5((s_1, s_2), \sigma)$ if

$(s_3, s_5) \in M_1(s_1, \sigma)$ and

$(s_4, s_6) \in M_2(s_2, \sigma)$.

$T(\mathfrak{A}_5) = T(\mathfrak{A}_1) \cap T(\mathfrak{A}_2)$.

$\square$

Note that 3-f.a.f.t. $\mathfrak{A}_6 = < S_1 \times S_2\ \Sigma, M_5, (s_{10}, s_{20}), F_6 >$, where S_1, S_2, Σ, M_5, s_{10}, and s_{20} are all as in the preceding proof, and $F_6 = (F_1 \times S_2) \cup (S_1 \times F_2)$, is not necessarily a union machine for $\mathfrak{A}_1$ and $\mathfrak{A}_2$. In general, $T(\mathfrak{A}_6) \supset T(\mathfrak{A}_1) \cup T(\mathfrak{A}_2)$, where $A \supset B$ indicates $A \supseteq B$ and $A \neq B$.

Note, also, that the cross product construction does yield a union machine for the leaf-up models. That is, given 1-f.a. f.t.'s (2-f.a.f.t.'s) $\mathfrak{A}_7 = < S_7, \Sigma, M_7, s_{70}, F_7 >$ and $\mathfrak{A}_8 = < S_8, \Sigma, M_8, s_{80}, F_8 >$. Define 1-f.a.f.t. (2-f.a.f.t., respectively) $\mathfrak{A}_9 = < S_7 \times S_8, \Sigma, M_9, (s_{70}, s_{80}), F_9 >$, where for all $s_1, s_3, s_5 \in S_7$, all $s_2, s_4, s_6 \in S_8$, and all $\sigma \in \Sigma$, $(s_1, s_2) \in M_9((s_3, s_4), (s_5, s_6), \sigma)$ if $s_1 \in M_7(s_3, s_5, \sigma)$ and $s_2 \in M_8(s_4, s_6, \sigma)$, and $F_9 = (S_7 \times F_8) \cup (F_7 \times S_8)$. $T(\mathfrak{A}_9) = T(\mathfrak{A}_7) \cup T(\mathfrak{A}_8)$.

The reader is urged to thoroughly consider the differences between root-down and leaf-up automata which the two preceding observations indicate.

Theorem 7: 3-f.a.f.t. and 4-f.a.f.t. are closed under cylindrification.

Proof: Given 3-f.a.f.t. (4-f.a.f.t.) $\mathfrak{A} = < S, \Sigma_1, M, s_0, F >$. Define 3-f.a.f.t. (4-f.a.f.t., respectively) $\mathfrak{A}_1 = < S, \Sigma_1 \times \Sigma_2, M_1, s_0, F >$, where for all $(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$, and all $s \in S$, $M_1(s, (\sigma_1, \sigma_2)) = M(s, \sigma_1)$. $T(\mathfrak{A}_1)$ = the Σ_2-cylindrification of $T(\mathfrak{A})$.

$\square$

<u>Claim 1</u>: $T(\mathfrak{A}_1) = Y_\Sigma - T(\mathfrak{A})$.

<u>Proof</u>: For each $s \in S$, define 3-f.a.f.t. $\mathfrak{A}_s = < S, \Sigma, M, s, F >$.

Consider a finite Σ-tree $e = (v,E)$.

First we state and prove Lemma 1, then we use Lemma 1 to show that if $e \notin T(\mathfrak{A})$ then $e \in T(\mathfrak{A}_1)$. Then we state Lemma 2 (which is the contrapositive of Lemma 1) and Lemma 3 (which is immediate from Lemma 2), and we use Lemma 3 to show that if $e \in T(\mathfrak{A})$ then $e \notin T(\mathfrak{A}_1)$.

Note that the following definition and Lemmas 1, 2, and 3 are all stated with respect to the finite Σ-tree e.

For $s \in S$ and $y \in E$, we will say that <u>Condition 1 holds for s at y</u> if $[\forall r \in Rn(\mathfrak{A}_s, (v, E \cap T_y))][r(Ft(E \cap T_y)) \nsubseteq F]$. For $\Delta \subseteq S$ and $y \in E$ we will say that Condition 1 holds for Δ at y iff for all $s \in \Delta$, Condition 1 holds for s at y.

<u>Lemma 1</u>: For all $s \in S$ and all $y \in E - Ft(E)$, if Condition 1 holds for s at y, then for all $(s_1, s_2) \in M(s, v(y))$, either 1) Condition 1 holds for s_1 at y0, or 2) Condition 1 holds for s_2 at y1, or both 1) and 2).

<u>Proof of Lemma 1</u>: Suppose Lemma 1 is false. Then for some $s \in S$ and some $y \in E - Ft(E)$, Condition 1 holds for s at y, and there exists $(s_1, s_2) \in M(s, v(y))$ such that Condition 1 does <u>not</u> hold for s_1 at y0 and Condition 1 does <u>not</u> hold for s_2 at y1. Hence, $[\exists r \in Rn(\mathfrak{A}_{s_1}, (v, E \cap T_{y0}))](r(Ft(E \cap T_{y0})) \subseteq F)$, and $[\exists r \in Rn(\mathfrak{A}_{s_2}, (v, E \cap T_{y1}))]$ $(r(Ft(E \cap T_{y1})) \subseteq F)$. But then clearly, $[\exists r \in Rn(\mathfrak{A}_s, (v, E \cap T_y))]$ $(r(Ft(E \cap T_y)) \subseteq F)$; and this contradicts the assumption that Condition 1 holds for s at y. Therefore, Lemma 1 is true.

We proceed with the proof of Claim 1.

Suppose $e \notin T(\mathfrak{A})$. We use Lemma 1 to construct an accepting $\mathfrak{A}_1$-run r_1 on e inductibely as follows.

Induction hypothesis:

1) For all $y \in E\text{-}Ft(E)$ such that $r_1(y)$ has been defined, we have Condition 1 holds for $r_1(y)$ at y.

2) For all $y \in E\text{-}Ft(E)$, a) $r_1(y0)$ has been defined iff $r_1(y1)$ has been defined, and b) if $r_1(y0)$ has been defined, then $(r_1(y0), r_1(y1)) \in M_1(r_1(y), v(y))$.

3) For all $y \in Ft(E)$, if $r_1(y)$ has been defined, then $r_1(y) \in F_1$.

Clearly, clauses 2) and 3) of the induction hypothesis will insure that r_1 is an accepting $\mathfrak{A}_1$-run on e.

Basis: $r_1(\Delta) = \{s_0\}$.

Since $e \notin T(\mathfrak{A})$, the induction hypothesis holds after the basis step.

Induction step: We assume r_1 is defined at $y \in E\text{-}Ft(E)$ and extend r_1 to y0 and y1 by defining $r_1(y0)$ and $r_1(y1)$ to be the sets containing only those states explicitly put into them by the following.

Case 1: $y0 \in Ft(E)$ and $y1 \notin Ft(E)$.

For all $(s_1, s_2) \in M(r_1(y), v(y))$, if Condition 1 holds for s_1 at y0, then put s_1 into $r_1(y0)$, else put s_2 into $r_1(y1)$.

Clearly from the definition of M_1, we have $(r_1(y0), r_1(y1)) \in M_1(r_1(y), v(y))$. Clearly from the construction of $r_1(y0)$, Condition 1 holds for $r_1(y0)$ at y0; and by clause 1) in the induction hypothesis and Lemma 1, Condition 1 holds for $r_1(y1)$ at y1.

Case 2: $y0 \in Ft(E)$ and $y1 \notin Ft(E)$.

For all $(s_1, s_2) \in M(r_1(y), v(y))$, if $s_1 \notin F$, then put s_1 into $r_1(y0)$, else put s_2 into $r_1(y1)$.

Clearly from the definition of M_1, we have $(r_1(y0), r_1(y1)) \in M_1(r_1(y), v(y))$. From the construction of $r_1(y0)$, we have $r_1(y0) \cap F = \phi$, and hence, from the definition of F_1 we have $r_1(y0) \in F_1$. For all $s \in F$, Condition 1 does not hold for s at y0. Hence, by clause 1) in the induction hypothesis and Lemma 1, we have Condition 1 holds for $r_1(y1)$ at y1.

Case 3: $y0 \notin Ft(E)$ and $y1 \in Ft(E)$.

Symmetric to Case 2.

Case 4: $y0 \in Ft(E)$ and $y1 \in Ft(E)$.

For all $(s_1, s_2) \in M(r_1(y), v(y))$, if $s_1 \notin F$, then put s_1 into $r_1(y0)$, else put s_2 into $r_1(y1)$.

Clearly from the definition of M_1, we have $(r_1(y0), r_1(y1) \in M_1(r_1(y), v(y))$. Suppose that for some $(s_1, s_2) \in M(r_1(y), v(y))$, we have $s_1 \in F$ and $s_2 \in F$. But then we have immediately that Condition 1 does not hold for some $s \in r_1(y)$ at y, and this contradicts the induction hypothesis. Hence, if $s_1 \in F$, then $s_2 \notin F$, and by construction $r_1(y0) \cap F = \emptyset$ and $r_1(y1) \cap F = \emptyset$. Hence, by the definition of F_1, we have $r_1(y0) \in F_1$ and $r_1(y1) \in F_1$.

This completes the induction.

By clauses 2) and 3) of the induction hypothesis r_1 is an accepting $\mathfrak{A}_1$-run on e. Therefore, if $e \notin T(\mathfrak{A})$, then $e \in T(\mathfrak{A}_1)$.

Lemma 2: For all $s \in S$ and all $y \in E\text{-}Ft(E)$, if Condition 1 does not hold for s at y, then for some $(s_1, s_2) \in M(s, v(y))$, Condition 1 does not hold for s_1 at y0, and Condition 1 does not hold for s_2 at y1.

Proof of Lemma 2: Lemma 2 is the contrpositive of Lemma 1.

Lemma 3: For all $r_1 \in Rn(\mathfrak{A}_1, e)$, and all $y \in E\text{-}Ft(E)$, if Condition 1 does not hold for $r_1(y)$ at y, then either 1) Condition 1 does not hold for $r_1(y0)$ at y0, or 2) Condition 1 does not hold for $r_1(y1)$ at y1, or both 1) and 2).

Proof of Lemma 3: Immediate from Lemma 2 and the definition of M_1.

We proceed with the proof of Claim 1.

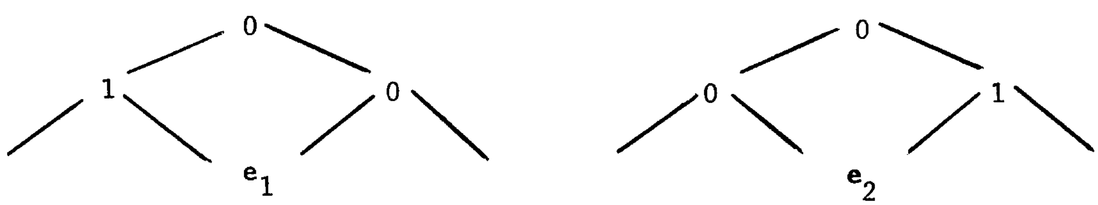
Suppose $e \in T(\mathfrak{A})$. Then Condition 1 does not hold for s_0 at Λ. Hence, by induction using Lemma 3, we have for all $r_1 \in Rn(\mathfrak{A}_1, (v,E))$ there exists a $y \in Ft(E)$ such that Condition 1 does not hold for $r_1(y)$ at y. That is, $r_1(y) \cap F \neq \emptyset$, and hence, $r_1(y) \notin F_1$, and r_1 is not an accepting $\mathfrak{A}_1$-run.

Therefore, if $e \in T(\mathfrak{A})$, then $e \notin T(\mathfrak{A}_1)$. This completes the proof of Claim 1. □

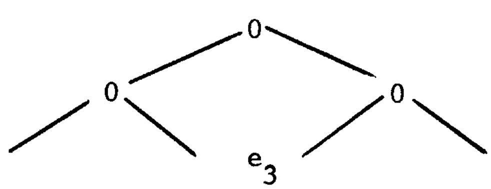
Theorem 9: 4-f.a.f.t. are not closed under union, projection, or complementation.

Proof: First we show that 4-f.a.f.t. are not closed under union.

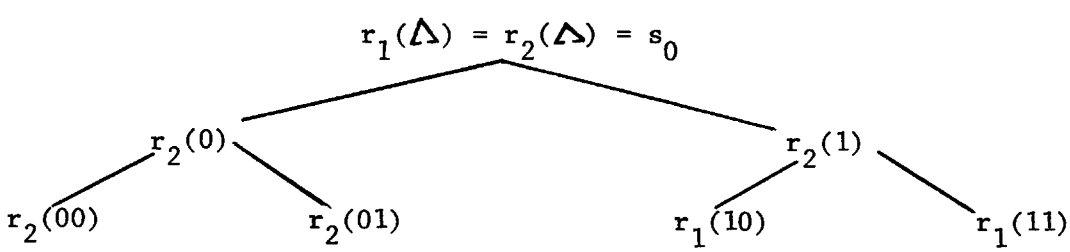
Let $B_2 = \{e_1, e_2\}$, where $e_1 = (v_1, E)$, $e_2 = (v_2, E)$, $E = \{\Lambda, 0, 1, 00, 01, 10, 11\}$, and v_1 and v_2 are given by the following pictures:



Suppose 4-f.a.f.t. $\mathfrak{A} = < S, \{0,1\}, M, s_0, F >$ defines B_2. Let r_1 and r_2 be the unique $\mathfrak{A}$-runs on e_1 and e_2, respectively. Consider $e_3 = (v_3, E)$ where v_3 is given by the following picture:



Clearly, the unique $\mathfrak{A}$-run on e_3 is given by the following picture:



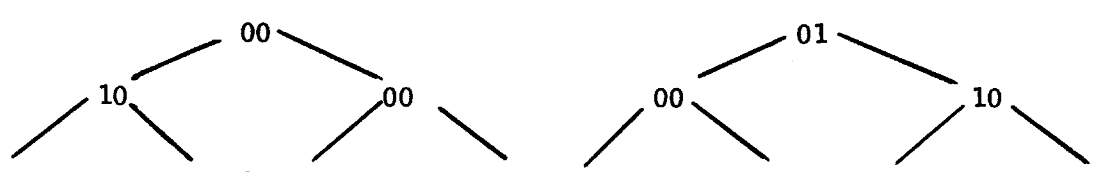
Since r_1 and r_2 are both accepting $\mathfrak{A}$-runs, we have $\{r_2(00), r_2(01), r_1(10), r_1(11)\} \subseteq F$, and $e_3 \in T(\mathfrak{A})$. But $e_3 \notin B_2$ and contrary to assumption $T(\mathfrak{A}) \neq B_2$. Hence B_2 is not 4-f.a.f.t. definable.

Let $B_3 = \{e_1\}$. B_3 is defined by 4-f.a.f.t. $\mathfrak{A}_1 = <\{s_0, s_1, s_2, f, R\}, \{0,1\}, M_1, s_0, \{f\}>$, where M_1 is given by the table:

M_1	0	1
s_0	(s_1, s_2)	(R, R)
s_1	(R, R)	(f, f)
s_2	(f, f)	(R, R)
f	(R, R)	(R, R)
R	(R, R)	(R, R) .

Let $B_4 = \{e_2\}$. B_4 is defined by a 4-f.a.f.t. symmetric $\mathfrak{A}_1$. $B_2 = B_3 \cup B_4$ is not 4-f.a.f.t. definable. Therefore, 4-f.a.f.t. are not closed under union.

Let $B_5 = \{e_4, e_5\}$, where $e_4 = (v_4, E)$, $e_5 = (v_5, E)$, E is as before, and v_4 and v_5 are given by the pictures:



e_4 e_5

B_5 is defined by 4-f.a.f.t. $\mathfrak{A}_2 = < \{s_0, s_1, s_2, f, R\}, \{0,1\}, M_2, s_0, \{f\} >$, where M_2 is given by the table:

M_2	00	01	10	11
s_0	(s_1, s_2)	(s_2, s_1)	(R, R)	(R, R)
s_1	(R, R)	(R, R)	(f, f)	(R, R)
s_2	(f, f)	(R, R)	(R, R)	(R, R)
f	(R, R)	(R, R)	(R, R)	(R, R)
R	(R, R)	(R, R)	(R, R)	(R, R) .

$B_2 = p_1B_5$ is not 4-f.a.f.t. definable. Therefore, 4-f.a.f.t. are not closed under projection.

Let $B_6 = \{(v_6, E_6) \in Y_{\{0,1\}} \mid (\forall x \in E_6\text{-}Ft(E_6))(v_6(x) = 0)\}$. Then $Y_{\{0,1\}}\text{-}B_6 = \{(v,E) \in Y_{\{0,1\}} \mid (\exists x \in E\text{-}Ft(E))(v(x) = 1)\}$. B_6 is defined by 4-f.a.f.t. $\mathfrak{A}_3 = < \{s_0, R\}, \{0,1\}, M_3, s_0, \{s_0\} >$, where M_3 is given by the table:

M_3	0	1
s_0	(s_0, s_0)	(R, R)
R	(R, R)	(R, R) .

Suppose 4-f.a.f.t. $\mathfrak{A}_4$ defines $Y_{\{0,1\}}\text{-}B_6$. $\{e_1, e_2\} \subseteq Y_{\{0,1\}}\text{-}B_6$ and hence there exist accepting $\mathfrak{A}_4$-runs on e_1 and e_2. By the same argument used for $\mathfrak{A}$, there must then be an accepting $\mathfrak{A}_4$-run on e_3, and $e_3 \in T(\mathfrak{A}_4)$. But $e_3 \in B_6$, and hence, contrary to our assumption $T(\mathfrak{A}_4) \neq Y_{\{0,1\}}\text{-}B_6$. Therefore, 4-f.a.f.t. are not closed under complementation. □

Theorem 10: 3-f.a.f.t. $\supset$ 4-f.a.f.t.

Proof: Every 4-f.a.f.t. is a 3-f.a.f.t.. Hence, we have immediately 3-f.a.f.t. $\supseteq$ 4-f.a.f.t.

Let the set B_2 be as defined in the previous proof. B_2 is defined by 3-f.a.f.t. $\mathfrak{A} = < \{s_0, s_1, s_2, f, R\}, \{0,1\}, M, s_0, \{f\} >$, where M is given by the table:

M	0	1
s_0	(s_1, s_2) (s_2, s_1)	(R, R)
s_1	(R, R)	(f, f)
s_2	(f, f)	(R, R)
f	(R, R)	(R, R)
R	(R, R)	(R, R) .

By the proof of Theorem 9, B_2 is not 4-f.a.f.t. definable. $\square$

Theorem 11: There exists a procedure which given any 3-f.a.f.t. with n states decides whether or not $T(\mathfrak{A}) = \phi$ in n^3 or fewer computational steps.

Proof: Given 3-f.a.f.t. on Σ-trees $\mathfrak{A} = < S, \Sigma, M, s_0, F >$, we first form the 3-f.a.f.t. on $\{0\}$-trees $\mathfrak{A}_1 = < S, \{0\}, M_1, s_0, F >$, where for all $s \in S$, $M_1(s, 0) = \bigcup_{\sigma\in\Sigma} M(s, \sigma)$. Clearly, $T(\mathfrak{A}) = \phi$ iff $T(\mathfrak{A}_1) = \phi$.

For each $s \in S$, define 3-f.a.f.t. $\mathfrak{A}_s = < S, \{0\}, M_1, s, F >$, where S, M_1, and F are as above. Let R denote the set of $s \in S$ such that there exists a finite tree $E \neq \{\Lambda\}$ and an $\mathfrak{A}_s$-run $r: E \rightarrow S$ such that $r(Ft(E)) \subseteq F$. (Remember that for every $\mathfrak{A}_s$-run r, we have $r(\Lambda) = s$.) We compute R recursively as follows:

1) $H_0 = \emptyset$,

2) for $i < \omega$, $H_{i+1} = H_i \cup \{s \mid (\exists s_1)(\exists s_2)[(s_1, s_2) \in M_1(s, 0), \{s_1, s_2\} \subseteq H_i \cup F]\}$.

$H_i \subseteq H_{i+1}$, for all $i < \omega$, and if $H_i = H_{i+1}$, then $H_i = H_{i+k} = R$, $k < \omega$. Since $H_i \subseteq S$, we are assured that $H_n = H_{n+1} = R$. Given H_i, the calculation of H_{i+1} requires at most n^2 steps since $c(H_i) \leq n$. That is, for each pair (s_1, s_2) such that $\{s_1, s_2\} \subseteq H_i \cup F$ we look at a previously constructed table to find all $s \in S$ such that $(s_1, s_2) \in M_1(s, 0)$. Hence, the calculation of R takes at most n^3 steps.

$T(\mathfrak{A}) \neq \emptyset$ iff $s_0 \in R$. □

The above procedure is an appropriate simplification of a procedure presented by Rabin in [10]. The informal notion of computational step used above is the one used by Rabin in [10].

In summary, the root-down nondeterministic finite tree automaton model is trivially equivalent to the leaf-up nondeterministic finite tree automaton model. The leaf-up nondeterministic finite tree automaton model is equivalent (by the subset machine construction) to the leaf-up deterministic finite tree automaton model. These automata models are all closed under union, intersection, complementation, projection and cylindrification. The root-down deterministic finite tree automaton model is strictly weaker than the above, and is closed only under intersection and cylindrification.

1-f.a.f.t. ≡ 2-f.a.f.t. ≡ 3-f.a.f.t. ⊃ 4-f.a.f.t.

	3-f.a.f.t.	4-f.a.f.t.
union	closed	no
intersection	closed	closed
complementation	closed	no
projection	closed	no
cylindrification	closed	closed

FIGURE 1

CHAPTER II

Finite Automata on Infinite Sequences

SECTION I INTRODUCTION

In 1960 Buchi [1] was the first to use finite automata on infinite sequences to obtain a decision procedure for a theory. This theory was the monadic second-order theory of one successor function. In 1966 McNaughton [5] proved his important fundamental result, the equivalence of the deterministic and nondeterministic variations of a finite automaton model on infinite input sequences.

As in Chapter I we do not state the immediate corollaries of each theorem. Instead, we summarize our theorems and their immediate corollaries in Figure 2 of section 7.

SECTION II DEFINITIONS

Definition: $T_1 = 1^*$.

The mapping ψ: $T_1 \to \mathbb{N}$ such that $1^n \mapsto n$ is a one-to-one correspondence between T_1 and $\mathbb{N}$. Hence, we sometimes use $\mathbb{N}$ for T_1 and $[\tau]$ for $\{\Lambda, 1, 11, \ldots, 1^{\tau-1}\}$, $\tau < \omega$.

Definition: A Σ^*-sequence on a finite alphabet Σ is a mapping w: $\{\Lambda, 1, 11, \ldots, 1^{\tau}\} \to \Sigma$, $\tau < \omega$ (or equivalently, w: $[\tau+1] \to \Sigma$). Σ^* is the set of all Σ^*-sequences. $\Sigma^+ = \Sigma^* - \{\lambda\}$.

<u>Definition</u>: A <u>Σ^{ω}-sequence</u> on a finite alphabet Σ is a mapping $v: T_1 \to \Sigma$ (or equivalently, $v: \mathbb{N} \to \Sigma$).

We formally defined both finite and infinite sequences as mappings. However, because it is very convenient to use concatenation, we will refer to finite and infinite strings of symbols on Σ as Σ^*-sequences and Σ^{ω}-sequences, respectively, and we will refer to Σ^*-sequences and Σ^{ω}-sequences as strings of symbols on Σ. For example, $x = 001$ is the $\{0,1\}^*$-sequences $x: [3] \to \{0,1\}$, where $x(0) = 0$, $x(1) = 0$, and $x(2) = 1$.

<u>Definition</u>: A regular event is any set $A \subseteq \Sigma^*$ which is finite automata definable.

We will denote a Σ^*-sequence by x, or y, or w, or x_i. Hence, if A is a set of finite strings on Σ, we use A^* to denote the set of all finite strings obtained by concatenating finitely many members of A. If $x, y \in \Sigma^*$, xy is the concatenation of x and y. We will denote a regular event by α, β, or γ. We will denote a symbol in the alphabet Σ by σ, or σ_i. We will denote a Σ^{ω}-sequence by v or v_i.

<u>Definition</u>: If $E \subseteq \Sigma^+$ then we denote by <u>E^{ω}</u> the set of all Σ^{ω}-sequences obtained by concatenating members of E infinitely many times. That is, $E^{\omega} = \{x_1 x_2 \ldots x_n \ldots \mid \text{for all } i \in \mathbb{N},\ x_i \in E\}$.

<u>Definition</u>: If $E \subseteq \Sigma^*$ and $F \subseteq \Sigma^+$, then the set <u>$E \cdot F^{\omega}$</u> $= \{x \cdot v \mid x \in E$ and $v \in F^{\omega}\}$.

Definition: A set $R \subseteq \Sigma^{\omega}$ is an ω-regular event if there exist regular events $E_1, \ldots, E_n, F_1, \ldots, F_n$, such that 1) $E_i \subseteq \Sigma^*$, and $F_i \subseteq \Sigma^+$, for all i, $1 \leq i \leq n$, and 2) $R = \bigcup_{i=1}^{n} E_i \cdot F_i^{\omega}$.

Definition: For a mapping $\psi: A \to B$, $In(\psi) = \{b \mid b \in B, c(\psi^{-1}(b)) \geq \omega\}$.

Definition: A mapping $r: \mathbb{N} \to s$ is 1-accepting with respect to $F \subseteq S$ if

$$(\exists t)\ r(t) \in F.$$

A mapping $r: \mathbb{N} \to S$ is 1'-accepting with respect to $F \subseteq S$ if

$$(\forall t)\ r(t) \in F.$$

A mapping $r: \mathbb{N} \to S$ is 2-accepting with $F \subseteq S$ if

$$In(r) \cap F \neq \emptyset.$$

A mapping $r: \mathbb{N} \to S$ is 2'-accepting with respect to $\mathcal{F} \subseteq P(S)$ if

$$(\exists F \in \mathcal{F})\ In(r) \subseteq F.$$

A mapping $r: \mathbb{N} \to S$ is 3-accepting with respect to $\mathcal{F} \subseteq P(S)$ if

$$In(r) \in \mathcal{F}.$$

A mapping $r: \mathbb{N} \to S$ is 4-accepting with respect to $\Omega = ((R_i, G_i))_{i<n}$, where for all $i < n$, $R_i \subseteq S$, $G_i \subseteq S$, if for some $i < n$,

$$In(r) \cap R_i = \emptyset \text{ and } In(r) \cap G_i \neq \emptyset.$$

In speaking it is often convenient to indicate that a run is 2-accepting by saying that it "accepts infinitely often", to indicate that a run is 2'-accepting by saying that it "eventually always accepts", and to indicate that a run is 3-accepting by saying that "the set of states entered infinitely often is a designated subset".

Definition: An n-table on Σ is a system $\mathfrak{M}' = < S, \Sigma, M, s_0 >$, where S is the finite state set, Σ is the finite alphabet set, $M: S \times \Sigma \to P(S) - \{\emptyset\}$ is the state transition function, and $s_0 \in S$ is the initial state.

Definition: A d-table on Σ is an n-table such that $M: S \times \Sigma \to \{\{s\} \mid s \in S\}$.

Definition: An $\mathfrak{M}'$-run on input w $\in \Sigma^*$, $w: [\tau] \to \Sigma$, $\tau < \omega$, is any mapping $r: [\tau+1] \to S$ such that 1) $r(0) = s_0$, and 2) for all $t < \tau$, $r(t+1) \in M(r(t), w(t))$.

Definition: For any $\tau \leq \omega$, any Σ^ω-sequence v, and any table $\mathfrak{M}' = < S, \Sigma, M, s_0 >$, a mapping $r: [\tau] \to S$ is called compatible with $\mathfrak{M}'$ and v if 1) $r(0) = s_0$, and 2) for all $t < \tau$, $r(t+1) \in M(r(t), v(t))$.

Definition: An $\mathfrak{M}'$-run on input v $\in \Sigma^\omega$ is any mapping $r: \mathbb{N} \to S$ which is compatible with table $\mathfrak{M}'$ and v.

We, also, talk about an $\mathfrak{M}$-run of an automaton $\mathfrak{M}$ on a sequence v(w) meaning an $\mathfrak{M}'$-run of the associated n(d)-table $\mathfrak{M}'$. The set of all $\mathfrak{M}$-runs on v(w) is denoted $Rn(\mathfrak{M}, v)$ $(Rn(\mathfrak{M}, w))$.

Nondeterministic (deterministic) finite automaton on finite sequences is abbreviated n.f.a.f. (d.f.a.f.).

Definition: An n.f.a.f. (d.f.a.f.) on Σ^* is a system $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), and $F \subseteq S$ is the set of accepting states.

f.a.f. $\mathfrak{M}$ accepts w: $[\tau] \to \Sigma$, $\tau < \omega$, if there exists an $\mathfrak{M}$-run r on w such that $r(\tau) \in F$.

The n.f.a.f. and d.f.a.f. are the familiar finite automaton models of conventional finite automata theory.

Nondeterministic (deterministic) finite automaton (on infinite sequences) is abbreviated n.f.a. (d.f.a.).

Definition: A 1-n.f.a. (1-d.f.a.) on Σ^ω is a system $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), and $F \subseteq S$ is the set of designated states.

1-f.a. $\mathfrak{M}$ accepts v if there exists an $\mathfrak{M}$-run on v which is 1-accepting with respect to F.

Definition: A 1'-n.f.a. (1'-d.f.a.) on Σ^ω is a system $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), and $F \subseteq S$ is the set of designated states.

1'-f.a. $\mathfrak{M}$ accepts v if there exists an $\mathfrak{M}$-run on v which is 1'-accepting with respect to F.

<u>Definition</u>: A <u>2-n.f.a.</u> <u>(2-d.f.a.)</u> on Σ^ω is a system $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), and $F \subseteq S$ is the set of <u>designated states</u>.

2-f.a. $\mathfrak{M}$ <u>accepts</u> v if there exists an $\mathfrak{M}$-run on v which is 2-accepting with respect to F.

<u>Definition</u>: A <u>2'-n.f.a.</u> <u>(2'-d.f.a.)</u> on Σ^ω is a system $\mathfrak{M} = < S, \Sigma, M, s_0, \mathcal{F} >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), and $\mathcal{F} \subseteq P(S)$ is the set of <u>designated subsets</u>.

2'-f.a. $\mathfrak{M}$ <u>accepts</u> v if there exists an $\mathfrak{M}$-run on v which is 2'-accepting with respect to $\mathcal{F}$.

<u>Definition</u>: A <u>3-n.f.a.</u> <u>(3-d.f.a.)</u> on Σ^ω is a system $\mathfrak{M} = < S, \Sigma, M, s_0, \mathcal{F} >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), and $\mathcal{F} \subseteq P(S)$ is the set of <u>designated subsets</u>.

3-f.a. $\mathfrak{M}$ <u>accepts</u> v if there exists an $\mathfrak{M}$-run on v which is 3-accepting with respect to $\mathcal{F}$.

<u>Definition</u>: A <u>4-n.f.a.</u> <u>(4-d.f.a.)</u> on Σ^ω is a system $\mathfrak{M} = < S, \Sigma, M, s_0, \Omega >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), and $\Omega = ((R_i, G_i))_{i<n}$, for all $i < n$, $R_i \subseteq S$, $G_i \subseteq S$, are the <u>subset pairs</u>.

4-f.a. $\mathfrak{M}$ <u>accepts</u> v if there exists an $\mathfrak{M}$-run on v which is 4-accepting with respect to Ω .

Definition: An t-th partial run of n-table (d-table) $\mathfrak{M}' = < S, \Sigma, M, s_0 >$ on v is any mapping r_t: $[t+1] \to S$ such that 1) $r_t(t) = s_0$, and 2) for all $i \in [t]$, $r_t(i) \in M(r_t(i+1), v(i))$.

Note that for any table $\mathfrak{M}'$ and any Σ^ω-sequence v, the 0-th partial run is the mapping $r_0(0) = s_0$, where s_0 is $\mathfrak{M}'$'s initial state.

Definition: A mapping p: $[\tau] \to S$, $\tau \le \omega$, is C-compatible with table $\mathfrak{M}'$ and Σ^ω-sequence v if for all $t \in [\tau]$, there exists r_t, a t-th partial $\mathfrak{M}'$-run on v, such that $r_t(0) = p(t)$.

Definition: A $\mathfrak{M}'$-C(compound)-run on v is any mapping r: $\mathbb{N} \to S$ which is C-compatible with $\mathfrak{M}'$ and v.

We will, also, speak of an i-th partial $\mathfrak{M}$-run and an $\mathfrak{M}$-C-run of a finite automaton $\mathfrak{M}$ meaning an i-th partial $\mathfrak{M}'$-run and an $\mathfrak{M}'$-C-run, respectively, of the associated n(d)-table $\mathfrak{M}'$. The set of all i-th parital runs of $\mathfrak{M}$ on v is denoted P-Rn($\mathfrak{M}$, v $|$ [i]). The set of all $\mathfrak{M}$-C-runs on v is denoted C-Rn($\mathfrak{M}$, v).

Definition: A 1C-n.f.a. (1C-d.f.a.) on Σ^ω is a system $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), and $F \subseteq S$ is the set of designated states.

1C-f.a. $\mathfrak{M}$ accepts v is there exists an $\mathfrak{M}$-C-run on v which is 1-accepting with respect to F.

Definition: A 1'C-n.f.a. (1'C-d.f.a.) on Σ^ω is a system $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), and $F \subseteq S$ is the set of designated states.

1'C-f.a. $\mathfrak{M}$ accepts v if there exists an $\mathfrak{M}$-C-run on v which is 1'-accepting with respect to F.

Definition: A 2C-n.f.a. (2C-d.f.a.) on Σ^{ω} is a system $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), and $F \subseteq S$ is the set of designated states.

2C-f.a. $\mathfrak{M}$ accepts v if there exists an $\mathfrak{M}$-C-run on v which is 2-accepting with respect to F.

Definition: A 2'C-n.f.a. (2'C-d.f.a.) on Σ^{ω} is a system $\mathfrak{M} = < S, \Sigma, M, s_0, \mathcal{F} >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), $\mathcal{F} \subseteq P(S)$ is the set of designated subsets.

2'C-f.a. $\mathfrak{M}$ accepts v if there exists an $\mathfrak{M}$-C-run on v which is 2'-accepting with respect to $\mathcal{F}$.

Definition: A 3C-n.f.a. (3C-d.f.a.) on Σ^{ω} is a system $\mathfrak{M} = < S, \Sigma, M, s_0, \mathcal{F} >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), and $\mathcal{F} \subseteq P(S)$ is the set of designated subsets.

3C-f.a. $\mathfrak{M}$ accepts v if there exists an $\mathfrak{M}$-C-run on v which is 3-accepting with respect to $\mathcal{F}$.

Definition: A 4C-n.f.a. (4C-d.f.a.) on Σ^{ω} is a system $\mathfrak{M} = < S, \Sigma, M, s_0, \Omega >$, where $< S, \Sigma, M, s_0 >$ is an n-table (d-table), and $\Omega = ((R_i, G_i))_{i<n}$, for all $i < n$, $R_i \subseteq S$, $G_i \subseteq S$, are the subsets pairs.

4C-f.a. $\mathfrak{M}$ accepts v if there exists an $\mathfrak{M}$-C-run on v which is 4-accepting with respect to Ω.

Definition: For mappings $M_1: A_1 \to B_1$ and $M_2: A_2 \to B_2$, the mapping $M_1 \times M_2: A_1 \times A_2 \to B_1 \times B_2$ is defined $M_1 \times M_2((a_1, a_2)) = (M_1(a_1), M_2(a_2))$, for all $(a_1, a_2) \in A_1 \times A_2$.

Definitions of $p_1(A)$ where $A \subseteq (\Sigma_1 \times \Sigma_2)^\omega$, Σ_2-cylindrification of $B \subseteq \Sigma_1^\omega$, $T(\mathfrak{M})$, i-f.a. definable, i-f.a. $\mathfrak{M}$ equivalent to j-f.a. $\mathfrak{M}_1$, i-f.a. closed under union, intersection, complementation, and projection, i-f.a. equivalent j-f.a., etc. are obtained by suitably modifying the corresponding definitions of Chapter I (i.e. by replacing (v,E) by v, and Y_Σ by Σ^ω).

As in the preceding definitions, we will write i-f.a. only where both i-n.f.a. and i-d.f.a. could be written. That is, where every occurrence of i-f.a. may be replaced by i-n.f.a., or every occurrence of i-f.a. may be replaced by i-d.f.a.

In [3] Hartmanis and Stearns study 1'd.f.a. In [4] Landweber investigates 1-d.f.a., 1'-d.f.a., 2-d.f.a., 2'-d.f.a., and 3-d.f.a. In [5] McNaughton uses 3-d.f.a. and 3-n.f.a., and the important consturction he uses to prove 3-d.f.a. ≡ 3-n.f.a. suggests the notion of a 4-accepting run. In [8] Rabin uses the notion of 4-accepting in his notion of dual acceptance, and in [7] Rabin uses the notion of 4-accepting run. In [1] Buchi uses 2-n.f.a.

SECTION II.I CONSEQUENCES OF $M: S \times \Sigma \rightarrow P(S) - \{\emptyset\}$

The preceding definitions specify that a nondeterministic automaton has a state transition function $M: S \times \Sigma \rightarrow P(S) - \{\emptyset\}$. The usual definition of a nondeterministic automaton allows all state transition functions $M: S \times \Sigma \rightarrow P(S)$. That is, the usual definition allows for there to exist state, input symbol pairs for which there are no transitions. Hence, there can be nondeterministic automata which have no runs on certain strings on their input alphabets. These strings are rejected because no accepting run on them exists. For finite automata on finite strings this causes no problems. In fact, by adding a non-accepting "trap" state one can easily obtain an equivalent finite automaton on finite strings with state transition function $M: S \times \Sigma \rightarrow P(S) - \{\emptyset\}$.

However, on infinite input sequences if all state transition functions $M: S \times \Sigma \rightarrow P(S)$ are allowed, then we get as a theorem 1-n.f.a. $\equiv$ 2'-n.f.a. In fact, we have the following construction.

Given 2'-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, F >$. Define 1-n.f.a. $\mathfrak{M}_1 = < S \cup F_1, \Sigma, M_1, s_0, F_1 >$, where $F_1 = F \times \{a\}$, for all $s \in S$, and all $\sigma \in \Sigma$, $M_1(s, \sigma) = M(s, \sigma) \cup \{(f,a) \mid f \in M(s, \sigma) \cap F\}$, and for all $(f,a) \in F_1$, and all $\sigma \in \Sigma$, $M_1((f,a), \sigma) = \{(f',a) \mid f' \in M(f, \sigma) \cap F\}$. We have $T(\mathfrak{M}) = T(\mathfrak{M}_1)$.

Note that the above construction uses state symbol pairs with no transitions in such a way that the essentially finite acceptance condition (of hitting an accepting state once) combined with the infinite condition of the existence of an infinite run in which this finite event occurs imply that there exists an accepting run of the 1-n.f.a. on v iff there exists a run of the 2'-n.f.a. on v which satisfys the infinite 2'-acceptance condition.

Our feeling that the above is an undesirable quirk is further reinforced when we see that the 1-d.f.a., and the 1'-, 2-, 2'-, and 3-n.f.a. models are not affected by allowing M: $S \times \Sigma \to P(S)$. That the 1-d.f.a. remains closed under projection even when 1-d.f.a. and 1-n.f.a. are not equivalent easily follows from the conventional subset construction for obtaining a deterministic automaton from a nondeterministic automaton.

Therefore, we defined an n-table as we did, and we have Lemma 2 below which we would not have if we had allowed M: $S \times \Sigma \to P(S)$. We state Lemmas 1 and 2 without proofs, because the proofs are trivial.

Lemma 1: For any mapping p: $[\tau] \to S$, $\tau \leq \omega$, compatible with d-table $\mathfrak{M}' = \langle S, \Sigma, M, s_0 \rangle$ and Σ^ω-sequence v, there exists a unique $r \in Rn(\mathfrak{M}', v)$ such that $r \mid [\tau] = p$. Hence, for $\mathfrak{M}'$, and all $v \in \Sigma^\omega$, we have $c(Rn(\mathfrak{M}', v)) = 1$.

For any mapping p: $[\tau] \to S$, $\tau \leq \omega$, C-compatible with $\mathfrak{M}'$ and $v \in \Sigma^\omega$, there exists a unique $r \in C\text{-}Rn(\mathfrak{M}', v)$ such that $r \mid [\tau] = p$. Hence, for $\mathfrak{M}'$ and all $v \in \Sigma^\omega$, we have $c(C\text{-}Rn(\mathfrak{M}', v)) = 1$.

Lemma 2: For any mapping $p: [\tau] \to S$, $\tau \leq \omega$, compatible with n-talbe $\mathfrak{M}' = < S, \Sigma, M, s_0 >$ and $v \in \Sigma^\omega$ there exists an $r \in Rn(\mathfrak{M}', v)$ such that $r \mid [\tau] = p$. Hence, for all $v \in \Sigma^\omega$, we have $Rn(\mathfrak{M}', v) \neq \phi$.

For any mapping $p: [\tau] \to S$, $\tau \leq \omega$, C-compatible with $\mathfrak{M}'$ and $v \in \Sigma^\omega$ there exists an $r \in C\text{-}Rn(\mathfrak{M}', v)$ such that $r \mid [\tau] = p$. Hence, for all $v \in \Sigma^\omega$, we have $C\text{-}Rn(\mathfrak{M}', v) \neq \phi$.

SECTION III USEFUL, INITIAL OBSERVATIONS

We can immediately make the following observations which we will use again and again in the following proofs.

Every d-table is an n-table so that we have for all $i \in \{1, 1', 2, 2', 3, 4, 1C, 1'C, 2C, 2'C, 3C, 4C\}$, i-d.f.a. $\subseteq$ i-n.f.a. We won't restate this in the following proofs, since we trust that the reader will detect when it is used without prompting.

Given 2'-f.a. (3-f.a., 2'C-f.a., 3C-f.a.) $\mathfrak{M} = < S, \Sigma, M, s_0, \{F_1, \ldots, F_k\} >$. For $1 \leq i \leq k$, define 2'-f.a. (3-f.a., 2'C-f.a., 3C-f.a., respectively) $\mathfrak{M}_i = < S, \Sigma, M, s_0, \{F_i\} >$. Clearly, $T(\mathfrak{M}) = T(\mathfrak{M}_1) \cup T(\mathfrak{M}_2) \cup \ldots \cup T(\mathfrak{M}_k)$. In Theorems 12 and 13 we show by construction directly from the definitions of 2-f.a., 2'-f.a., 3-f.a., 2'C-f.a., and 3C-f.a. that they are all closed under union. On the basis of these observations several of the following proofs are stated for 2'-f.a. (3-f.a., 2'C-f.a., 3C-f.a.) with a single designated subset. We trust that the reader will realize that the generalization indicated above is implied in such cases.

<u>Lemma 3</u>: Given any 1-f.a. on Σ^{ω} $\mathfrak{M}$ we can determine an equivalent 1-f.a. $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, \{f\} >$ such that $s_{10} \neq f$, and for all $\sigma \in \Sigma$, $M_1(f, \sigma) = \{f\}$.

Given any 1C-f.a. on Σ^{ω} $\mathfrak{M}_2$ we can determine an equivalent 1C-f.a. $\mathfrak{M}_3 = < S_3, \Sigma, M_3, s_{30}, F_3 >$ such that $s_{30} \notin F_3$.

<u>Proof</u>: Immediate from Lemmas 1 and 2 and the definition of 1-f.a. and 1C-f.a. Note that if $\mathfrak{M}$ ($\mathfrak{M}_2$) is deterministic, then we can make $\mathfrak{M}_1$ ($\mathfrak{M}_3$, respectively) deterministic. □

<u>Lemma 4</u>: Given any 1'-f.a. on Σ^{ω} $\mathfrak{M}$, we can determine an equivalent 1'-f.a. $\mathfrak{M}_1 = < F_1 \cup \{s\}, \Sigma, M_1, s_{10}, F_1 >$ such that $s_{10} \in F_1$, and for all $\sigma \in \Sigma$, $M_1(s, \sigma) = \{s\}$.

Given any 1'C-f.a. on Σ^{ω} $\mathfrak{M}_2$, we can determine an equivalent 1'C-f.a. $\mathfrak{M}_3 = < S_3, \Sigma, M_3, s_{30}, F_3 >$ such that $s_{30} \in F_3$.

<u>Proof</u>: Immediate from Lemmas 1 and 2 and the definitions of 1'-f.a. and 1'C-f.a. Note that if $\mathfrak{M}$ ($\mathfrak{M}_2$) is deterministic, then we can make $\mathfrak{M}_1$ ($\mathfrak{M}_3$, respectively) deterministic. □

SECTION IV EQUIVALENCE OF MODELS

Our primary intention is to present a clear exposition which, hopefully, will speed the beginner's acquisition of a facile, intuitive grasp of various notions of a finite automaton running on an infinite sequence. Hence, our presentation involves some redundancy. Note in particular that many closure properties are proven for both i-n.f.a. and i-d.f.a., after we have proven that i-n.f.a. ≡ i-d.f.a. We do this because the constructions used work equally well for n.f.a. and d.f.a., and we see no reason to hide this sometimes useful fact. However, in order to avoid uninformative redundancy, we begin by showing that many models are equivalent. Then given a property which we wish to show our models have (or fail to have), we can give the proof for the model which has the simplest, most informative proof and we get as corollaries that all equivalent models have the property (fail to have the property). In fact, in some cases (for example, closure of 1'C-f.a., 2'C-f.a., 3C-f.a. under projection) we do not know how to prove more directly what follows easily from the equivalence of models.

Theorem 1: 1-n.f.a. ≡ 1-d.f.a.

Proof: Given 1-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, F >$. Define 1-d.f.a. $\mathfrak{M}_1 = < P(S), \Sigma, M_1, \{s_0\}, F_1 >$, where for all $\Delta \in P(S)$, and all $\sigma \in \Sigma$, $M_1(\Delta, \sigma) = \{ \bigcup_{s \in \Delta} M(s, \sigma)\}$, and $F_1 = \{\Delta \in P(S) \mid \Delta \cap F \neq \emptyset\}$.

Suppose $v \in T(\mathfrak{M})$. Then there exist $r \in Rn(\mathfrak{M}, v)$ and $t \in \mathbb{N}$ such that $r(t) \in F$. By the construction of $\mathfrak{M}_1$, if r_1 is the unique $\mathfrak{M}_1$-run on v, then $r(t) \in r_1(t)$. Hence, $r_1(t) \in F_1$, and $v \in T(\mathfrak{M}_1)$.

Suppose $v \in T(\mathfrak{M}_1)$. Then there exists $r_1 \in Rn(\mathfrak{M}_1, v)$ and $t \in \mathbb{N}$ such that $r_1(t) \in F_1$. By the construction of $\mathfrak{M}_1$, there exists a mapping $p\colon [t+1] \to S$ compatible with $\mathfrak{M}$ and v and such that $p(t) \in F$. Hence, by Lemma 2, there exists $r \in Rn(\mathfrak{M}, v)$ such that $r \mid [t+1] = p$. Hence, $r(t) \in F$ and $v \in T(\mathfrak{M})$.

Therefore, $T(\mathfrak{M}) = T(\mathfrak{M}_1)$, and 1-n.f.a. $\subseteq$ 1-d.f.a. □

<u>Theorem 2</u>: 1'-n.f.a. $\equiv$ 1'-d.f.a.

<u>Proof</u>: By Lemma 4, given any 1'-n.f.a. $\mathfrak{M}$ we can determine an equivalent 1'-n.f.a. $\mathfrak{M}_1 = < F_1 \cup \{s\}, \Sigma, M_1, s_{10}, F_1 >$, where $s_{10} \in F_1$, and for all $\sigma \in \Sigma$, $M_1(s, \sigma) = \{s\}$.

Define 1'-d.f.a. $\mathfrak{M}_2 = < P(F_1), \Sigma, M_2, \{s_{10}\}, F_2 >$, where for all $\Delta \in P(F_1)$, for all $\sigma \in \Sigma$, $M_2(\Delta, \sigma) = \{ \bigcup_{s \in \Delta} M_1(s, \sigma)\}$, and $F_2 = P(F_1) - \{\emptyset\}$ [illegible]

Clearly, because the construction of $\mathfrak{M}_1$ eliminated all transitions from s into F_1, the unique $\mathfrak{M}_2$-run r_2 on v is such that for all $t \in \mathbb{N}$, $r_2(t) \in F_2$ iff there is an $\mathfrak{M}_1$-run r_1 on v such that $r_1([t]) \subseteq F_1$. Hence, $r_2(\mathbb{N}) \subseteq F_2$ iff there exists $r_1 \in Rn(\mathfrak{M}_1, v)$ such that $r_1(\mathbb{N}) \subseteq F_1$.

Therefore, $T(\mathfrak{M}) = T(\mathfrak{M}_1)$ and 1'-n.f.a. $\subseteq$ 1'-d.f.a. □

<u>Theorem 3</u>: 2'-n.f.a. ≡ 2'-d.f.a.

<u>Proof</u>: (This construction was suggested to me by A.R. Meyer in private communication.)

Given 2'-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, \{F\} >$. Define 2'-d.f.a. $\mathfrak{M}_1 =$ $< S_1, \Sigma, M_1, s_{10}, \mathcal{F}_1 >$, where $S_1 = P(S) \times P(F)$,

$$s_{10} = \begin{cases} (\{s_0\}, \{s_0\}), & \text{if } s_0 \in F, \\ (\{s_0\}, \emptyset), & \text{else,} \end{cases}$$

for all $(D, A) \in S_1$, and all $\sigma \in \Sigma$,

$M_1((D, A), \sigma) =$ (D', A'), where $D' = M(D, \sigma)$ and

(I) if $A = \emptyset$ then $A' = D' \cap F$,

(II) if $A \neq \emptyset$ then $A' = M(A, \sigma) \cap F$; and $\mathcal{F}_1 = \{\{(D, A) \mid A \neq \emptyset\}\}$.

Suppose $v \in T(\mathfrak{M}_1)$. Then for the unique $\mathfrak{M}_1$-run r_1 on v we have $In(r_1) \subseteq F_1$. Hence, $(\exists\tau)(\forall t)(\tau \leq t \rightarrow r_1(t) \in F_1)$; and by (II) in the definition of M_1, there exists $r \in Rn(\mathfrak{M}, v)$ such that for all t, $\tau \leq t \rightarrow r(t) \in F$. Hence, $In(r) \subseteq F$ and $v \in T(\mathfrak{M})$.

Suppose $v \in T(\mathfrak{M})$. Then there exists $r \in Rn(\mathfrak{M}, v)$ such that $In(r) \subseteq F$. Hence,

(III) $(\exists\tau)(\forall t)(\tau \leq t \rightarrow r(t) \in F)$.

Suppose there exists $t_1 \in \mathbb{N}$ such that

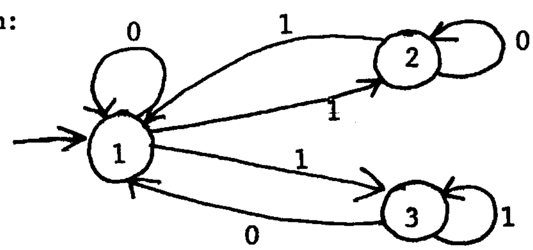
(IV) $(\tau < t_1 \ \& \ r_1(t_1) \notin F_1)$, where r_1 is the unique $\mathfrak{M}_1$-run on v.

That is, such that $r_1(t_1) = (D, \emptyset)$, some $D \subseteq S$. Let $r_1(t_1+1) = (D', A')$. By our construction of $\mathfrak{M}_1$ we have $r(t_1+1) \in D'$. Hence, by (III) we have $D' \cap F \neq \emptyset$. Hence, by (I) in the definition of M_1, $A' = D' \cap F \neq \emptyset$. Then by (II) in the definition of M_1, and (III) above, we have for all t, $t_1 + 1 \leq t \rightarrow r_1(t) \in F_1$. Hence, $In(r_1) \subseteq F_1$, and $v \in T(\mathfrak{M}_1)$.

Therefore, $T(\mathfrak{M}) = T(\mathfrak{M}_1)$ and 2'-n.f.a. $\subseteq$ 2'-d.f.a. □

The above proof shows that there can be at most one time t_1 defined as above. The following is a 2-n.f.a. $\mathfrak{M}$, and the 2'-d.f.a. $\mathfrak{M}_1$ obtained from $\mathfrak{M}$ by the construction in the proof above.

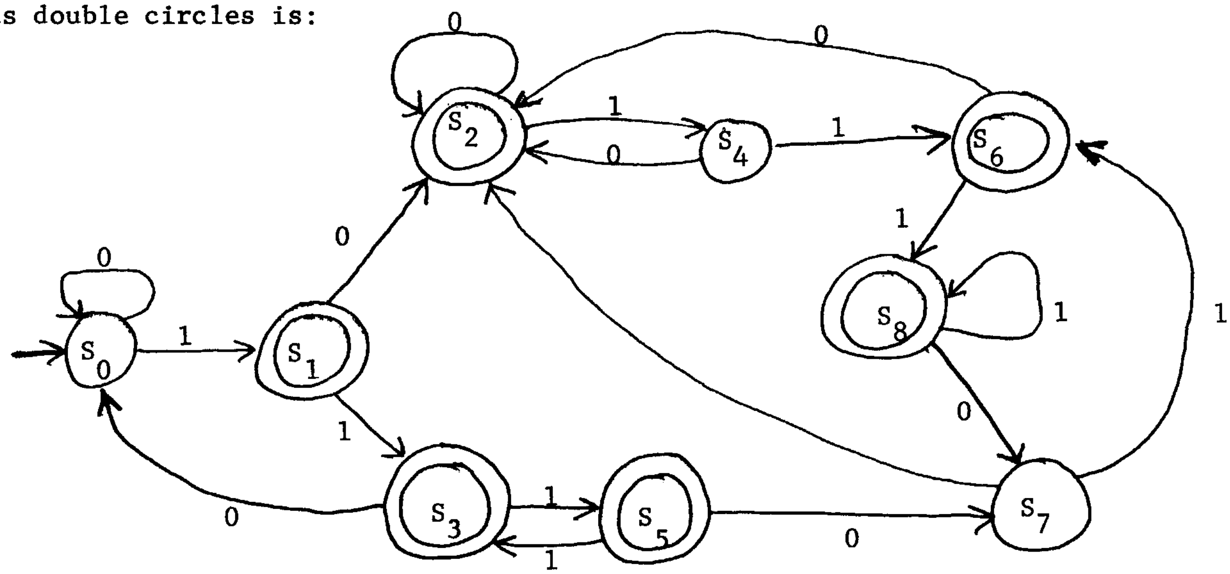
2'-n.f.a. $\mathfrak{M} = \langle \{1, 2, 3\}, \{0,1\}, M, 1, \{\{2, 3\}\} \rangle$, where M is given by the diagram:



Looking at the diagram for M we can construct the state transition table for M_1 (For convenience we use parentheses instead of set brackets.):

	0	1
$s_0 = ((1), \emptyset)$	s_0	s_1
$s_1 = ((2, 3), (2, 3))$	s_2	s_3
$s_2 = ((1, 2), (2))$	s_2	s_4
$s_3 = ((1, 3), (3))$	s_0	s_5
$s_4 = ((1, 2, 3), (\emptyset))$	s_2	s_6
$s_5 = ((2, 3), (3))$	s_7	s_3
$s_6 = ((1, 2, 3), (2, 3))$	s_2	s_8
$s_7 = ((1, 2), \emptyset)$	s_2	s_6
$s_8 = ((1, 2, 3), (3))$	s_7	s_8

2'-d.f.a. $\mathfrak{M}_1 = < \{s_0, \ldots, s_8\}, \{0,1\}, M_1, s_0, \{\{s_1, s_2, s_3, s_5, s_6, s_8\}\} >$, where M_1 is given by the table above. The state transition diagram for M_1 with the members of the designated subset of $\mathfrak{M}_1$ marked as double circles is:



$r = 1313^{\omega}$ is an accepting $\mathfrak{M}$-run on 101^{ω}. $r_1 = s_0s_1s_2s_4s_6s_8^{\omega}$ is the accepting $\mathfrak{M}_1$-run on 101^{ω}. Note that $(\forall t)(3 \leq t \rightarrow r(t) \in F)$, where F is the designated subset of $\mathfrak{M}$. But $r_1(4) = s_4$ is not in the designated subset of $\mathfrak{M}_1$. Hence, a time t_1 as defined in the proof of Theorem 3 does exist (i.e. $t_1 = 4$) for the $\mathfrak{M}_1$-run on 101^{ω}.

Theorem 4: 2-n.f.a. $\equiv$ 3-n.f.a.

Proof: Given 2-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, F >$. Define 3-n.f.a. $\mathfrak{M}_1 = < S, \Sigma, M, s_0, \mathcal{F}_1 >$, where $\mathcal{F}_1 = \{F_1 \subseteq S \mid F_1 \cap F \neq \emptyset\}$. Clearly, $T(\mathfrak{M}_1) = T(\mathfrak{M})$.

Given 3-n.f.a. $\mathfrak{M}_2 = < S_2, \Sigma, M_2, s_{20}, \{F_2\} >$. Define 2-n.f.a. $\mathfrak{M}_3 = < S_3, \Sigma, M_3, s_{20}, F_3 >$, where $S_3 = S_2 \cup (F_2 \times P(F_2)) \cup \{s_T\}$, for all $s \in S_2$, and all $\sigma \in \Sigma$, $M_3(s, \sigma) \supseteq M_2(s, \sigma)$; for all $s \in S_2$, and all $\sigma \in \Sigma$, if $f \in M_2(s, \sigma) \cap F_2$, then $(f, \phi) \in M_3(s, \sigma)$; for all $f \in F_2$, all $\sigma \in \Sigma$, all $D \subseteq F_2$, and all $f_1 \in M_2(f, \sigma) \cap F_2$, $s_T \in M_3((f, D), \sigma)$, if $D \neq F_2$ then $(f_1, D \cup \{f\}) \in M_3((f, D), \sigma)$, if $D = F_2$ then $(f_1, \{f\}) \in M_3((f, D), \sigma)$; for all $\sigma \in \Sigma$, $s_T \in M_3(s_T, \sigma)$; and $F_3 = F_2 \times \{F_2\}$.

Suppose $v \in T(\mathfrak{M}_2)$. Then there exists $r_2 \in Rn(\mathfrak{M}_2, v)$ such that $In(r_2) = F_2$. Therefore, there exist $t_0 < t_1 < t_2 \ldots$, such that $(\forall t)(t_0 < t \rightarrow r_2(t) \in F_2)$, and for all $i \in \mathbb{N}$, $r_2(\{t \mid t_i < t < t_{i+1}\}) \subset F_2$, and $r_2(\{t \mid t_i < t \leq t_{i+1}\}) = F_2$. From the definition of M_3, there exists $r_3 \in Rn(\mathfrak{M}_3, v)$ such that $r_3(\{t \mid t \leq t_0\}) \subseteq S_2$, $r_3(t_0 + 1) = (r_2(t_0 + 1), \phi)$, and for all $i > 0$, $r_3(t_i + 1) = (r_2(t_i + 1), F_2) \in F_3$. F_3 is finite, hence, $In(r_3) \cap F_3 \neq \phi$ and $v \in T(\mathfrak{M}_3)$.

Suppose $v \in T(\mathfrak{M}_3)$. Then there exists $r_3 \in Rn(\mathfrak{M}_3, v)$ such that $In(r_3) \cap F_3 \neq \phi$. From the definition of M_3, $p_1 r_3 \in Rn(\mathfrak{M}_2, v)$. We easily see the $p_1 r_3$ is an accepting $\mathfrak{M}_2$-run on v as follows. There exists some $(f, F_2) \in F_3$, and $t_0 < t_1 < t_2 \ldots$, such that for all $i \in \mathbb{N}$, $r_3(t_i) = (f, F_2)$. From the definition of M_3 for all $i \in \mathbb{N}$, $p_1 r_3(\{t \mid t_i \leq t < t_{i+1}\}) = F_2$. Hence, $In(p_1 r_3) = F_2$, $p_1 r_3$ is an accepting $\mathfrak{M}_2$-run on v, and $v \in T(\mathfrak{M}_2)$. □

<u>Theorem 5</u>: 3-n.f.a. ≡ 3-d.f.a.

<u>Proof</u>: This is McNaughton's important fundamental result in [5]. McNaughton proved the following theorem.

Theorem: A set $A \subseteq \Sigma^{\omega}$ is an ω-regular event iff A is 3-d.f.a. definable. (We will denote the content of this theorem by 3-d.f.a. ≡ ω-regular.) Actually McNaughton also shows that every 3-n.f.a. definable set is an ω-regular event. This coupled with the fact that every 3-d.f.a. is also a 3-n.f.a. means that his construction of a 3-d.f.a. to accept an arbitrary ω-regular event suffices to show 3-n.f.a. ≡ 3-d.f.a. □

<u>Alternative proof</u>: By McNaughton [5] 3-d.f.a. ≡ ω-regular. By Theorem 23 2-n.f.a. ≡ ω-regular. By Theorem 4 3-n.f.a. ≡ 2-n.f.a. Therefore, 3-n.f.a. ≡ 3-d.f.a. □

<u>Theorem 6</u>: 4-n.f.a. ≡ 3-n.f.a. ~~and 4-d.f.a.~~ ≡ 3-d.f.a.

<u>Proof</u>: Given 4-f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, ((R_i, G_i))_{i \le n} >$. For each i, $0 \le i \le n$, define 4-f.a. $\mathfrak{M}_i = < S, \Sigma, M, s_0, ((R_i, G_i)) >$. Clearly, $T(\mathfrak{M}) = T(\mathfrak{M}_1) \cup \ldots \cup T(\mathfrak{M}_n)$. By Theorem 12 both 3-f.a. and 4-f.a. are closed under union, hence the following suffices.

Given 4-f.a. $\mathfrak{M} = < S, \Sigma, s_0, ((R, G)) >$. Define 3-f.a. $\mathfrak{M}_1 = < S, \Sigma, M, s_0, \mathcal{F}_1 >$, where $\mathcal{F}_1 = \{F \subseteq S \mid F \cap R = \emptyset \ \& \ F \cap G \neq \emptyset\}$. Clearly, $T(\mathfrak{M}) = T(\mathfrak{M}_1)$. Hence, 4-f.a. ⊆ 3-f.a.

Given 3-f.a. $\mathfrak{M}_2 = < S_2, \Sigma, M_2, s_{20}\ \{F_2\} >$. Define 4-f.a. $\mathfrak{M}_3 = < S_2 \times P(F_2), \Sigma, M_3, (s_{20}, \emptyset), \Omega_3 >$, where for all $s \in S_2$, all $f \subseteq F_2$, and all $\sigma \in \Sigma$, if $f \neq F_2$ then $M_3((s, f), \sigma) = \{(s', f') \mid f' = (f \cup \{s\}) \cap F_2$, and $s' \in M_2(s, \sigma)\}$, if $f = F_2$ then $M_3((s, f), \sigma) = \{(s', \emptyset) \mid s' \in M_2(s, \sigma)\}$, and $\Omega_3 = ((\{(s, f) \in S_2 \times P(F_2) \mid s \notin F_2\}, \{(s, F_2) \mid s \in S_2\}))$.

Note that if $\mathfrak{M}_2$ is deterministic then $\mathfrak{M}_3$ is deterministic.

Clearly the above construction is very similar to the construction used in the proof of Theorem 4. Having seen this proof we trust the reader can easily complete the proof that $T(\mathfrak{M}_2) = T(\mathfrak{M}_3)$. Hence, 3-f.a. $\subseteq$ 4-f.a.

$\square$

SECTION IV.1 THE C-RUN MODELS

Professor A.R. Meyer suggested the definition of C-run to me in private communication. The notion behind C-run is clearly that of successively starting the finite automaton further and further down the infinite sequence, running it back to the beginning of the sequence, and noting its final state. Then basing acceptance on the sequence of final states obtained in this manner. A little thought, perhaps, is necessary to convince oneself that the process described above is <u>not</u> equivalent to noting the sequence of final states obtained by successively running the finite automaton from the beginning of the sequence to the end of longer and longer initial segments of the infinite sequence. This latter process is clearly just the notion of run used by 1-, 1'-, ..., 4-f.a. . Thus, perhaps, the following results are slightly surprising.

<u>Theorem 7</u>: 1C-n.f.a. ≡ 1C-d.f.a., 1'C-n.f.a. ≡ 1'C-d.f.a.,

2C-n.f.a. ≡ 2C-d.f.a., 2'C-n.f.a. ≡ 2'C-d.f.a.,

3C-n.f.a. ≡ 3C-d.f.a., 4C-n.f.a. ≡ 4C-d.f.a.

<u>Proof</u>: The following proof may be summed up by saying that the subset construction works for this model because the possible $\mathfrak{M}$-C-runs on v are determined by the possible <u>final</u> states, $r_i(0)$, for $r_i \in$ P-Rn($\mathfrak{M}$, v | [i]), i ∈ $\mathbb{N}$.

Given an n-table $\mathfrak{M}' = < S, \Sigma, M, s_0 >$. Define a d-table $\mathfrak{M}_1' = < P(S), \Sigma, M_1, \{s_0\} >$, where for all $\Delta \in P(S)$, and all $\sigma \in \Sigma$, $M_1(\Delta,\sigma) = \{ \bigcup_{s\in} M(s,\sigma)\}$. (Note that $\mathfrak{M}_1'$ is constructed from $\mathfrak{M}'$ by the standard subset construction of conventional finite automata theory.)

Clearly, for the unique $r_{1t} \in$ P-Rn($\mathfrak{M}_1'$, $v \mid [t]$), we have $(\forall r_t \in$ P-Rn($\mathfrak{M}'$, $v \mid [t]))(r_t(0) \in r_{1t}(0))$, and $(\forall s \in r_{1t}(0))$ $(\exists r_t \in$ P-Rn($\mathfrak{M}'$, $v \mid [t]))(r_t(o) = s)$. Hence for the unique $r_1 \in$ C-Rn($\mathfrak{M}_1'$, v), we have for all $t \in \mathbb{N}$,

(I) $r_1(t) = \{r(t) \mid r \in$ C-Rn($\mathfrak{M}'$, v)$\} \neq \phi$, and

(II) C-Rn($\mathfrak{M}'$, v) $= \{r: \mathbb{N} \to S \mid$ for all $t \in \mathbb{N}$, $r(t) \in r_1(t)\} \neq \phi$.

Given 1C-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 > = \mathfrak{M}'$. Define 1C-d.f.a. $\mathfrak{M}_1 = < P(S), \Sigma, M_1, \{s_0\}, F_1 >$, where $< P(S), \Sigma, M_1, \{s_0\} > = \mathfrak{M}_1'$, and $F_1 = \{\Delta \subseteq S \mid \Delta \cap F \neq \phi\}$.

Suppose $v \in T(\mathfrak{M})$. Then there exists $r \in$ C-Rn($\mathfrak{M}$, v) and $t \in \mathbb{N}$ such that $r(t) \in F$. By (I) and the definition of F_1, $r(t) \in r_1(t) \in F_1$, where r_1 is the unique $\mathfrak{M}_1$-C-run on v. Hence, $v \in T(\mathfrak{M}_1)$.

Suppose $v \in T(\mathfrak{M}_1)$. Then for the unique $r_1 \in$ C-Rn($\mathfrak{M}_1$, v) there exists a $t \in \mathbb{N}$ such that $r_1(t) \in F_1$. By the definition of F_1, $r_1(t) \cap F \neq \phi$, hence by (II), there exists $r \in$ C-Rn($\mathfrak{M}$, v) such that $r(t) \in F$. Hence, $v \in T(\mathfrak{M})$.

Therefore, $T(\mathfrak{M}) = T(\mathfrak{M}_1)$, and 1C-d.f.a. $\equiv$ 1C-n.f.a.

Given 1'C-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 > = \mathfrak{M}'$. Define 1'C-d.f.a. $\mathfrak{M}_1 = < P(S), \Sigma, M_1, \{s_0\}, F_1 >$, where $< P(S), \Sigma, M_1, \{s_0\} > = \mathfrak{M}_1'$, and $F_1 = \{\Delta \subseteq S \mid \Delta \cap F \neq \phi\}$.

Suppose $v \in T(\mathfrak{M})$. Then there exists $r \in C\text{-}Rn(\mathfrak{M}, v)$ such that $r(\mathbb{N}) \subseteq F$. By (I), for all $t \in \mathbb{N}$, $r(t) \in r_1(t)$ and hence, $r_1(t) \cap F = \phi$, where r_1 is the unique $\mathfrak{M}_1$-run on v. By the definition of F_1, $r_1(\mathbb{N}) \subseteq F_1$. Hence, $v \in T(\mathfrak{M}_1)$.

Suppose $v \in T(\mathfrak{M}_1)$. Then for the unique $r_1 \in C\text{-}Rn(\mathfrak{M}_1, v)$ we have $r_1(\mathbb{N}) \subseteq F_1$. Hence, for all $t \in \mathbb{N}$, $r_1(t) \cap F \neq \phi$. Hence by (II), there exists $r \in C\text{-}Rn(\mathfrak{M}, v)$ such that $R(\mathbb{N}) \subseteq F$. Hence, $v \in T(\mathfrak{M})$.

Therefore, $T(\mathfrak{M}) = T(\mathfrak{M}_1)$ and 1'C-n.f.a. $\equiv$ 1'C-d.f.a.

Given 2C-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 > = \mathfrak{M}'$. Define 2C-d.f.a. $\mathfrak{M}_1 = < P(S), \Sigma, M_1, \{s_0\}, F_1 >$, where $< P(S), \Sigma, M_1, \{s_0\} > = \mathfrak{M}_1'$, and $F_1 = \{\Delta \subseteq S \mid \Delta \cap F \neq \phi\}$.

Suppose $v \in T(\mathfrak{M})$. Then there exists $r \in C\text{-}Rn(\mathfrak{M}, v)$ such that $In(r) \cap F \neq \phi$. That is, there exist $t_0 < t_1 < \ldots$, such that for all $i \in \mathbb{N}$, $r(t_i) \in F$, hence by (I) and the definition of F_1, $r_1(t_i) \in F_1$, where r_1 is the unique $\mathfrak{M}_1$-C-run on v. Hence, $In(r_1) \cap F_1 \neq \phi$ and $v \in T(\mathfrak{M}_1)$.

Suppose $v \in T(\mathfrak{M}_1)$. Then for the unique $r_1 \in C\text{-}Rn(\mathfrak{M}_1, v)$ we have $In(r_1) \cap F_1 \neq \phi$. That is, there exist $t_0 < t_1 < \ldots$, such that for all $i \in \mathbb{N}$, $r_1(t_i) \in F_1$, and hence by the definition of F_1, $r_1(t_i) \cap F \neq \phi$, and by (II), there exists $r \in C\text{-}Rn(\mathfrak{M}, v)$ such that for all $i \in \mathbb{N}$, $r(t_i) \in F$. Hence, $v \in T(\mathfrak{M})$.

Therefore, $T(\mathfrak{M}) = T(\mathfrak{M}_1)$, and 2C-n.f.a. ≡ 2C-d.f.a.

Given 2'C-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, \{F\} >$, where $< S, \Sigma, M, s_0 > = \mathfrak{M}'$. Define 2'C-d.f.a. $\mathfrak{M}_1 = < P(S), \Sigma, M_1, \{s_0\}, \{F_1\} >$, where $< P(S), \Sigma, M_1, \{s_0\} > = \mathfrak{M}_1'$, and $F_1 = \{\Delta \subseteq S \mid \Delta \cap F \neq \emptyset\}$.

Suppose $v \in T(\mathfrak{M})$. Then there exists $r \in$ C-Rn$(\mathfrak{M}, v)$ such that $In(r) \subseteq F$. Again by (I) and the definition of F_1, $In(r_1) \subseteq F_1$, where r_1 is the unique $\mathfrak{M}_1$-C-run on v. Hence, $v \in T(\mathfrak{M}_1)$.

Suppose $v \in T(\mathfrak{M}_1)$. Then for the unique $r_1 \in$ C-Rn$(\mathfrak{M}_1, v)$ we have $In(r_1) \subseteq F_1$. Again by (II) and the definition of F_1, we have the existence of an $r \in$ C-Rn$(\mathfrak{M}, v)$ such that $In(r) \subseteq F$. Hence, $v \in T(\mathfrak{M})$.

Therefore, $T(\mathfrak{M}) = T(\mathfrak{M}_1)$, and 2'C-n.f.a. ≡ 2'C-d.f.a.

Given 3C-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, \{F\} >$, where $< S, \Sigma, M, s_0 > = \mathfrak{M}'$, and $F = \{s_1, s_2, \ldots, s_k\}$. Define 3C-d.f.a. $\mathfrak{M}_1 = < P(S), \Sigma, M_1, \{s_0\}, \mathcal{F}_1 >$, where $< P(S), \Sigma, M_1, \{s_0\} > = \mathfrak{M}_1'$, and $\mathcal{F}_1 = \{F_1 \subseteq P(S) \mid (\forall \Delta \in F_1)(\Delta \cap F \neq \emptyset) \;\&\; (\forall s \in F)(\exists \Delta \in F_1)(s \in \Delta)\}$.

Suppose $v \in T(\mathfrak{M})$. Then there exists $r \in$ C-Rn$(\mathfrak{M}, v)$ such that $In(r) = F$. Hence by (I), for all $\Delta \in In(r_1)$, $\Delta \cap F \neq \emptyset$, and for all $s \in F$ there exists $\Delta \in In(r_1)$ such that $s \in \Delta$, where r_1 is the unqiue $\mathfrak{M}_1$-C-run on v. By the definition of $\mathcal{F}_1$, $In(r_1) \in \mathcal{F}_1$, and hence, $v \in T(\mathfrak{M}_1)$.

Suppose $v \in T(\mathfrak{M}_1)$. Then for the unique $r_1 \in$ C-Rn$(\mathfrak{M}_1, v)$ we have $In(r_1) \in \mathcal{F}_1$. Hence, by the definition of $\mathcal{F}_1$, there exist $t_0 < t_1 < \ldots$, such that for all $t > t_0$, $r_1(t) \cap F \neq \emptyset$, and for all $1 \leq j \leq k$, and all $i \in \mathbb{N}$, $s_j \in r_1(t_{j+ki})$. Hence, by (II) there exists $r \in$ C-Rn$(\mathfrak{M}, v)$ such that $In(r) = F$, and $v \in T(\mathfrak{M})$.

Therefore, $T(\mathfrak{M}) = T(\mathfrak{M}_1)$, and 3C-n.f.a. ≡ 3C-d.f.a.

Given 4C-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, ((R, G)) >$, where $< S, \Sigma, M, s_0 > = \mathfrak{M}'$. Define 4C-d.f.a. $\mathfrak{M}_1 = < P(S), \Sigma, M_1, \{s_0\}, ((R_1, G_1)) >$, where $< P(S), \Sigma, M_1, \{s_0\} > = \mathfrak{M}_1'$, $R_1 = \{\mathcal{A} \subseteq S \mid \mathcal{A} \subseteq R\}$, and $G_1 = \{\mathcal{A} \subseteq S \mid \mathcal{A} \cap (G-R) \neq \emptyset\}$.

Suppose $v \in T(\mathfrak{M})$. Then there exists $r \in C\text{-}Rn(\mathfrak{M}, v)$ such that $In(r) \cap R = \emptyset$ and $In(r) \cap G \neq \emptyset$. Hence by (I) and the definitions of R_1 and G_1, $In(r_1) \cap R_1 = \emptyset$ and $In(r_1) \cap G_1 \neq \emptyset$, where r_1 is the unique $\mathfrak{M}_1$-C-run on v. Hence, $v \in T(\mathfrak{M}_1)$.

Suppose $v \in T(\mathfrak{M}_1)$. Then for the unique $r_1 \in C\text{-}Rn(\mathfrak{M}_1, v)$ we have $In(r_1) \cap R_1 = \emptyset$ and $In(r_1) \cap G_1 \neq \emptyset$. Hence there exist $t_0 < t_1 < \ldots$, such that for all $t > t_0$, $r_1(t) \cap (S-R) \neq \emptyset$, and for all $i \in \mathcal{N}$, $r_1(t_i) \cap (G-R) \neq \emptyset$. Hence by (II), there exists $r \in C\text{-}Rn(\mathfrak{M}, v)$ such that $In(r) \cap R = \emptyset$ and $In(r) \cap G \neq \emptyset$. Hence, $v \in T(\mathfrak{M})$.

Therefore, $T(\mathfrak{M}) = T(\mathfrak{M}_1)$, and 4C-n.f.a. ≡ 4C-d.f.a. □

Lemma 5:

1-d.f.a. ⊆ 1C-n.f.a., 1'-d.f.a. ⊆ 1'C-n.f.a.,

2-d.f.a. ⊆ 2C-n.f.a., 2'-d.f.a. ⊆ 2'C-n.f.a.,

3-d.f.a. ⊆ 3C-n.f.a., 4-d.f.a. ⊆ 4C-n.f.a.

Proof: Given d-table $\mathfrak{M}' = < S, \Sigma, M, s_0 >$. Define n-table $\mathfrak{M}_1' = < S_1, \Sigma, M_1, s_{10} >$, where $S_1 = (S \times S) \cup \{s_{10}, s_T\}$, for all $\sigma \in \Sigma$, $M_1(s_{10}, \sigma) = \{(s_1, s_2) \in (S \times S) \mid s_2 \in M(s_1, \sigma)\} \cup \{s_T\}$, for all $(s_1, s_2) \in (S \times S)$, and all $\sigma \in \Sigma$, $M_1((s_1, s_2), \sigma) = \{s_1', s_2) \in (S \times S) \mid s_1 \in M(s_1', \sigma)\} \cup \{s_T\}$, and for all $\sigma \in \Sigma$, $M_1(s_T, \sigma) = \{s_T\}$.

Clearly, for all $t > 0$ there exists $r_t \in P\text{-}Rn(\mathfrak{M}_1, v \mid [t])$ such that $r_t(0) = (s_0, s)$ iff for the unique $r \in Rn(\mathfrak{M}, v)$ we have $r(t) = s$. Thus,

(I) for all $t > 0$ there exists $r_1 \in C\text{-}Rn(\mathfrak{M}_1, v)$ such that $r_1(t) = (s_0, s)$ iff for the unique $r \in Rn(\mathfrak{M}, v)$ we have $r(t) = s$; and there exists $r_1 \in C\text{-}Rn(\mathfrak{M}_1, v)$ such that for all $t > 0$,

(II) $r_1(t) = (s_0, r(t))$, where r is the unique $\mathfrak{M}$-run on v.

Given 1-d.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 > = \mathfrak{M}'$. By Lemma 3 we assume without loss of generality that $s_0 \notin F$. Define 1C-n.f.a. $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, F_1 >$, where $< S_1, \Sigma, M_1, s_{10} > = \mathfrak{M}_1'$, and $F_1 = \{(s_0, s) \in S_1 \mid s \in F\}$.

Clearly from (I), $T(\mathfrak{M}) = T(\mathfrak{M}_1)$. Thus 1-d.f.a. $\subseteq$ 1C-n.f.a.

Given 1'-d.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 > = \mathfrak{M}'$. By Lemma 4 we assume without loss of generality that $s_0 \in F$. Define 1'C-n.f.a. $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, F_1 >$, where $< S_1, \Sigma, M_1, s_{10} > = \mathfrak{M}_1'$, and $F_1 = \{(s_0, s) \in (S \times S) \mid s \in F\} \cup \{s_{10}\}$. Clearly, from (I) $T(\mathfrak{M}) \supseteq T(\mathfrak{M}_1)$, and from (II) $T(\mathfrak{M}) \subseteq T(\mathfrak{M}_1)$. Thus, 1'-d.f.a. $\subseteq$ 1'C-n.f.a.

Given 2-d.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, F >$, where $< S, \Sigma, M, s_0 > = \mathfrak{M}'$. Define 2C-n.f.a. $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, F_1 >$, where $< S_1, \Sigma, M_1, s_{10} > = \mathfrak{M}_1'$, and $F_1 = \{(s_0, s) \in (S \times S) \mid s \in F\}$. Clearly from (I), $T(\mathfrak{M}) \supseteq T(\mathfrak{M}_1)$, and from (II), $T(\mathfrak{M}) \subseteq T(\mathfrak{M}_1)$. Thus 2-d.f.a. $\equiv$ 2C-n.f.a.

Given 2'-d.f.a. (3-d.f.a.) $\mathfrak{M} = < S, \Sigma, M, s_0, \mathcal{F} >$, where $< S, \Sigma, M, s_0 > = \mathfrak{M}'$. Define 2'C-n.f.a. (3C-n.f.a., respectively) $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, \mathcal{F}_1 >$, where $< S_1, \Sigma, M_1, s_{10} > = \mathfrak{M}_1'$, and $\mathcal{F}_1 = \{F \subseteq (S \times S) \mid p_1(F) = \{s_0\}$, and $p_2(F) \in \mathcal{F}\}$. Clearly from (I), $T(\mathfrak{M}) \supseteq T(\mathfrak{M}_1)$, and from (II), $T(\mathfrak{M}) \subseteq T(\mathfrak{M}_1)$. Thus, 2'-d.f.a. $\subseteq$ 2'C-n.f.a. and 3-d.f.a. $\subseteq$ 3C-n.f.a.

Given 4-d.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, \Omega >$, where $< S, \Sigma, M, s_0 > = \mathfrak{M}'$, and $\Omega = ((R_i, G_i))_{i<n}$. Define 4C-n.f.a. $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, \Omega_1 >$, where $< S_1, \Sigma, M_1, s_{10} > = \mathfrak{M}_1'$, and $\Omega_1 = ((R_{1i}, G_{1i}))$, where for all $i < n$, $R_{1i} = \{(s_1, s_2) \in (S \times S) \mid s_1 \neq s_0$ or $s_2 \in R_i\}$, and $G_{1i} = \{(s_o, s) \in (S \times S) \mid s \in G_i\}$. Clearly from (I), $T(\mathfrak{M}) \supseteq T(\mathfrak{M}_1)$, and from (II), $T(\mathfrak{M}) \subseteq T(\mathfrak{M}_1)$. Thus 4-d.f.a. $\subseteq$ 4C-n.f.a. □

<u>Lemma 6</u>:

1C-n.f.a. $\subseteq$ 1-d.f.a., 1'C-n.f.a. $\subseteq$ 1'-d.f.a.

2C-n.f.a. $\subseteq$ 2-d.f.a., 2'C-n.f.a. $\subseteq$ 2'-d.f.a.,

3C-n.f.a. $\subseteq$ 3-d.f.a., 4C-n.f.a. $\subseteq$ 4-d.f.a.

<u>Proof</u>: Given n-table $\mathfrak{M}' = < S, \Sigma, M, s_0 >$. Define d-table $\mathfrak{M}_1' = < S_1, \Sigma, M_1, s_{10} >$, where $S_1 = P(S \times S) \cup \{s_{10}\}$, for all $\sigma \in \Sigma$, $M_1(s_{10}, \sigma) = \{\{(s_1, s_2) \in (S \times S) \mid s_2 \in M(s_1, \sigma)\}\}$, and for all $\Delta \in P(S \times S)$, and all $\sigma \in \Sigma$, $M_1(\Delta, \sigma) = \{\{(s_1', s_2) \in (S \times S) \mid (\exists (s_1, s_2) \in \Delta)(s_1 \in M(s_1', \sigma)\}\}$.

By the definition of $\mathfrak{M}_1'$ there exists a unique $r_1 \in Rn(\mathfrak{M}_1', v)$ such that for all $t > 0$, there exists $r_t \in P\text{-}Rn(\mathfrak{M}', v \mid [t]))$ such that $r_t(0) = s$ iff $(s_0, s) \in r_1(t)$. By Lemmas 1 and 2 for all $t \in \mathbb{N}$, $P\text{-}Rn(\mathfrak{M}', v \mid [t]) \neq \phi$. Hence, for all $t \in \mathbb{N}$, $r_1(t) \neq \phi$. Hence,

(I) $\quad C\text{-}Rn(\mathfrak{M}', v) = \{r: \mathbb{N} \to S \mid r(0) = s_0$, and for all $t > 0$, $(s_0, r(t)) \in r_1(t)\}$, where r_1 is the unique $\mathfrak{M}_1'$-run on v.

Given 1C-n.f.a. $\mathfrak{M} = \langle S, \Sigma, M, s_0, F \rangle$, where $\langle S, \Sigma, M, s_0 \rangle = \mathfrak{M}'$. By Lemma 3 we assume without loss of generality that $s_0 \notin F$. Define 1-d.f.a. $\mathfrak{M}_1 = \langle S_1, \Sigma, M_1, s_{10}, F_1 \rangle$, where $\langle S_1, \Sigma, M_1, s_{10} \rangle = \mathfrak{M}_1'$, and $F_1 = \{\Delta \in P(S \times S) \mid \Delta \cap (\{s_0\} \times F) \neq \emptyset\}$. Clearly from (I), $T(\mathfrak{M}) \supseteq T(\mathfrak{M}_1)$, and from (I) and $s_0 \notin F$, $T(\mathfrak{M}) \subseteq T(\mathfrak{M}_1)$. Thus, 1C-n.f.a. $\subseteq$ 1-d.f.a.

Given 1'C-n.f.a. $\mathfrak{M} = \langle S, \Sigma, M, s_0, F \rangle$, where $\langle S, \Sigma, M, s_0 \rangle = \mathfrak{M}'$. By Lemma 4 we assume without loss of generality that $s_0 \in F$. Define 1'd-f.a. $\mathfrak{M}_1 = \langle S_1, \Sigma, M_1, s_{10}, F_1 \rangle$, where $\langle S_1, \Sigma, M_1, s_{10} \rangle = \mathfrak{M}_1'$, and $F_1 = \{\Delta \in P(S \times S) \mid \Delta \cap (\{s_0\} \times F) \neq \emptyset\} \cup \{s_{10}\}$. Clearly from (I) and $s_0 \in F$, $T(\mathfrak{M}) \supseteq T(\mathfrak{M}_1)$, and from (I), $T(\mathfrak{M}) \subseteq T(\mathfrak{M}_1)$. Thus, 1'C-n.f.a. $\subseteq$ 1'-d.f.a.

Given 2C-n.f.a. $\mathfrak{M} = \langle S, \Sigma, M, s_0, F \rangle$, where $\langle S, \Sigma, M, s_0 \rangle = \mathfrak{M}'$. Define 2-d.f.a. $\mathfrak{M}_1 = \langle S_1, \Sigma, M_1, s_{10}, F_1 \rangle$, where $\langle S_1, \Sigma, M_1, s_{10} \rangle = \mathfrak{M}_1'$, and $F_1 = \{\Delta \in P(S \times S) \mid \Delta \cap (\{s_0\} \times F) \neq \emptyset\}$. Clearly from (I), $T(\mathfrak{M}) = T(\mathfrak{M}_1)$. Thus, 2C-n.f.a. $\subseteq$ 2-d.f.a.

Given 2'C-n.f.a. (3C-n.f.a.) $\mathfrak{M} = \langle S, \Sigma, M, s_0, \mathcal{F} \rangle$, where $\langle S, \Sigma, M, s_0 \rangle = \mathfrak{M}'$. Define 2-d.f.a. (3-d.f.a., respectively) $\mathfrak{M}_1 = \langle S_1, \Sigma, M_1, s_{10}, \mathcal{F}_1 \rangle$, where $\langle S_1, \Sigma, M_1, s_{10} \rangle = \mathfrak{M}_1'$, and $\mathcal{F}_1 = \{F_1 \subseteq P(S \times S) \mid (\exists F \in \mathcal{F})(\forall \Delta \in F_1)(\Delta \cap (\{s_0\} \times F) \neq \emptyset)\ \&\ (\forall s \in F)(\exists \Delta \in F_1)((s_0, s) \in \Delta)\}$. Clearly from (I), $T(\mathfrak{M}) = T(\mathfrak{M}_1)$. Thus, 2'C-n.f.a. $\subseteq$ 2'-d.f.a. and 3C-n.f.a. $\subseteq$ 3-d.f.a.

Given 4C-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, \Omega >$, where $< S, \Sigma, M, s_0 > = \mathfrak{M}'$, and $\Omega = ((R_i, G_i))_{i<n}$. Define 4-d.f.a. $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, \Omega_1 >$, where $< S_1, \Sigma, M_1, s_{10} > = \mathfrak{M}_1'$, and $\Omega_1 = ((R_{1i}, G_{1i}))_{i<n}$, where $R_{1i} = \{\Delta \in P(S \times S) \mid \Delta \cap (\{s_0\} \times (S-R_i) = \emptyset\}$, and $G_{1i} = \{\Delta \in P(S \times S) \mid \Delta \cap (\{s_0\} \times (G_i - R_i) \neq \emptyset\}$.

Suppose $v \in T(\mathfrak{M})$. Then there exists $r \in C\text{-}Rn(\mathfrak{M}, v)$ such that for some $i < n$, $In(r) \cap R_i = \emptyset$ and $In(r) \cap G_i \neq \emptyset$. Hence, there exist $t_0 < t_1 < t_2 \ldots$, such that for all $t > t_0$, $r(t) \notin R_i$ and for all $j \in \mathbb{N}$, $r(t_j) \in G_i$. Thus by (I) for all $t > t_0$, $r_1(t) \notin R_{1i}$, and for all $j \in \mathbb{N}$, $r_1(t_j) \in G_{1i}$, where r_1 is the unique $\mathfrak{M}_1$-run on v. Hence, $In(r_1) \cap R_{1i} = \emptyset$ and $In(r_1) \cap G_{1i} \neq \emptyset$. Hence, $v \in T(\mathfrak{M}_1)$.

Suppose $v \in T(\mathfrak{M}_1)$. Then for the unique $r_1 \in Rn(\mathfrak{M}_1, v)$ we have for some $i < n$, $In(r_1) \cap R_{1i} = \emptyset$ and $In(r_1) \cap G_{1i} \neq \emptyset$. Hence there exist $t_0 < t_1 < t_2 \ldots$, such that for all $t > t_0$, $r_1(t) \notin R_{1i}$, and for all $j \in \mathbb{N}$, $r_1(t_j) \in G_{1i}$. Thus by (I), there exists $r \in C\text{-}Rn(\mathfrak{M}, v)$ such that for all $t > t_0$, $r(t) \notin R_i$, and for all $j \in \mathbb{N}$, $r(t_j) \in G_i$. Hence, $In(r) \cap R_i = \emptyset$ and $In(r) \cap G_i \neq \emptyset$. Hence, $v \in T(\mathfrak{M})$.

Therefore, $T(\mathfrak{M}) = T(\mathfrak{M}_1)$, and 4C-n.f.a. $\subseteq$ 4-d.f.a. □

<u>Theorem 8</u>:

1C-f.a. ≡ 1-f.a.,	1'C-f.a. ≡ 1'f.a.,
2C-f.a. ≡ 2-d.f.a.,	2'C-f.a. ≡ 2'-f.a.,
3C-f.a. ≡ 3-f.a.,	4C-f.a. ≡ 4-f.a.

Proof: Immediate from Lemmas 5 and 6, 1-n.f.a. ≡ 1-d.f.a. (Theorem 1), 1'-n.f.a. ≡ 1'd-f.a. (Theorem 2), 2'-n.f.a. ≡ 2'-d.f.a. (Theorem 3), 3-n.f.a. ≡ 3-d.f.a. (Theorem 5), 4-f.a. ≡ 3-f.a. (Theorem 6), and Theorem 7. □

SECTION V CLOSURE PROPERTIES

We now show that for certain i ∈ {1, 1', 2, ..., 4C} there exist procedures which given an i-f.a. $\mathfrak{M}$ (i-f.a.'s $\mathfrak{M}_1$ and $\mathfrak{M}_2$) yield an i-f.a. $\mathfrak{M}_3$ which defines the complement (projection, union, etc.) of the set(s) defined by $\mathfrak{M}$ ($\mathfrak{M}_1$ and $\mathfrak{M}_2$). For example, the reader will have attained a good understanding of the proof that 1-n.f.a. are closed under projection when he thinks something to the effect, "Of course, this is exactly how you should change machine $\mathfrak{M}$ in order to obtain machine $\mathfrak{M}_1$ such that $T(\mathfrak{M}_1) = p_2(T(\mathfrak{M}))$."

We trust that the reader can immediately see how to transform any i-f.a. (i ∈ {1, 1', 2, ..., 4C}) on Σ_1^ω $\mathfrak{M}$ into an i-f.a. $(\Sigma_1 \times \Sigma_2)^\omega$ $\mathfrak{M}_1$ such that $T(\mathfrak{M}_1) = \Sigma_2$-cylindrification of $T(\mathfrak{M})$. Hence, we claim without further proof that all of our models are closed under cylindrification.

Theorem 9: 1-, 1'-, 2-, 3-n.f.a. are closed under projection.

Proof: Given n-table $\mathfrak{M}' = \langle S, \Sigma_1 \times \Sigma_2, M, s_0 \rangle$. Define n-table $\mathfrak{M}_1' = \langle S, \Sigma_2, M_1, s_0 \rangle$, where for all $s \in S$, and all $\sigma_2 \in \Sigma_2$, $M_1(s, \sigma_2) = \bigcup_{\sigma_1 \in \Sigma_1} M(s, (\sigma_1, \sigma_2))$.

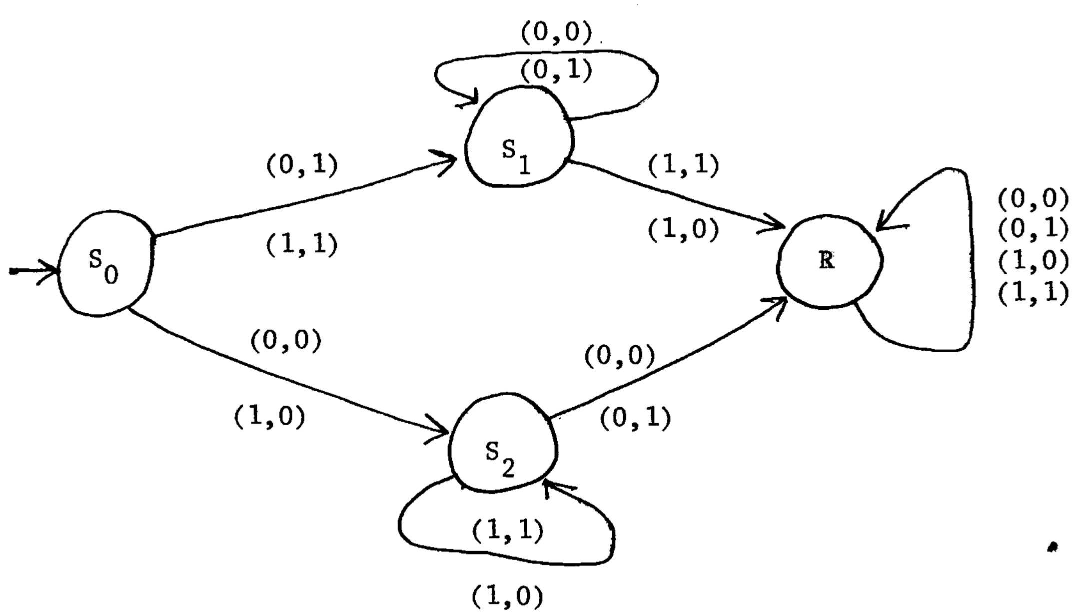
Let $v \in (\Sigma_1 \times \Sigma_2)^\omega$. For each $r \in Rn(\mathfrak{M}', v)$ there exists

(I) $r_1 \in Rn(\mathfrak{M}_1', p_2 v)$ such that for all $t \in \mathbb{N}$, $r(t) = r_1(t)$.

(II) Let $v_2 \in \Sigma_2^\omega$. For each $r_1 \in Rn(\mathfrak{M}_1', v_2)$ there exists $v \in (\Sigma_1 \times \Sigma_2)^\omega$, $p_2 v = v_2$, and $r \in Rn(\mathfrak{M}', v)$ such that for all $t \in \mathbb{N}$, $r(t) = r_1(t)$.

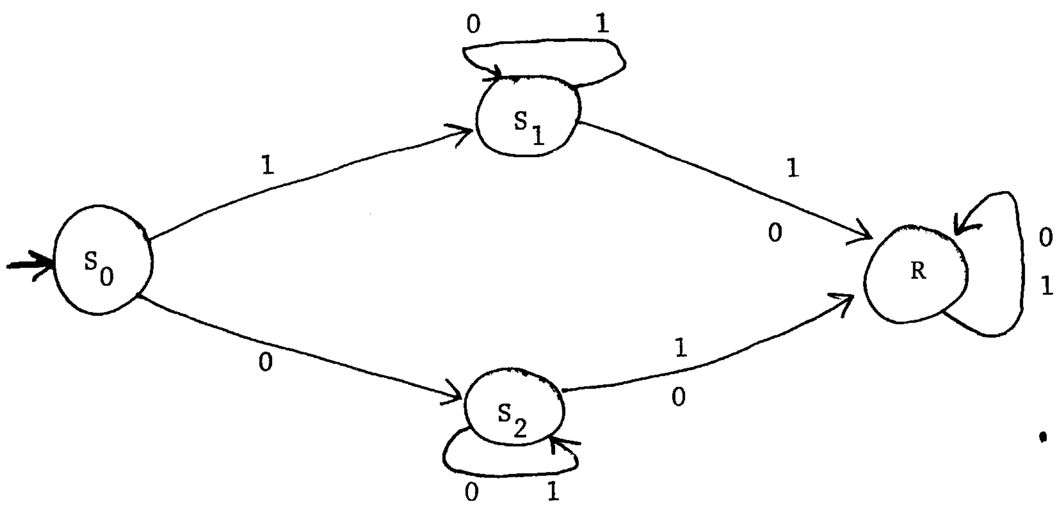
Given 1-n.f.a. (1'-n.f.a., 2-n.f.a.) $\mathfrak{M} = < S, \Sigma_1 \times \Sigma_2, M, s_0, F >$, and 2'-n.f.a. (3-n.f.a.) $\mathfrak{M}_2 = < S, \Sigma_1 \times \Sigma_2, M, s_0, \mathcal{F} >$, where $< S, \Sigma_1 \times \Sigma_2, M, s_0 > = \mathfrak{M}'$. Define 1-n.f.a. (1'-n.f.a., 2-n.f.a., respectively) $\mathfrak{M}_1 = < S, \Sigma_2, M_1, s_0, F >$, and 2'-n.f.a. (3-n.f.a., respectively) $\mathfrak{M}_3 = < S, \Sigma_2, M_1, s_0, \mathcal{F} >$, where $< S, \Sigma_2, M_1, s_0 > = \mathfrak{M}_1'$. From (I) we have $p_2(T(\mathfrak{M})) \subseteq T(\mathfrak{M}_1)$ and $p_2(T(\mathfrak{M}_2)) \subseteq T(\mathfrak{M}_3)$. From (II) we have $p_2(T(\mathfrak{M})) \supseteq T(\mathfrak{M}_1)$ and $p_2(T(\mathfrak{M}_2)) \supseteq T(\mathfrak{M}_3)$. □

It is interesting to note that the above construction does not work for 1C-, ..., 4C-n.f.a. For example, given 2'C-d.f.a. $\mathfrak{M} = < \{s_0, s_1, s_2, R\}, \{0, 1\}^2, M, s_0, \{\{s_1, s_2\}\} >$, where M is given by the diagram:



$T(\mathfrak{M}) = \{v \mid (p_2(In(v)) = \{0\}\ \&\ p_1(v(/\!\!N)) = \{1\})$ or $(p_2(In(v)) = \{1\}\ \&\ p_1(v(/\!\!N)) = \{0\})\}$.

The construction in the preceeding proof yields 2'C-n.f.a. $\mathfrak{M}_1$ with state transition diagram:



$p_2(T(\mathfrak{M})) = \{v \mid In(v) = \{0\}$ or $In(v) = \{1\}\}$. $T(\mathfrak{M}_1) = \{0, 1\}^\omega \neq p_2(T(\mathfrak{M}))$.

<u>Theorem 10</u>: 1-f.a. are closed under union and intersection.

<u>Proof</u>: Given 1-f.a.'s $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, F_1 >$ and $\mathfrak{M}_2 = < S_2, \Sigma, M_2, s_{20}, F_2 >$.

Define 1-f.a. $\mathfrak{M}_3 = < S_1 \times S_2, \Sigma, M_1 \times M_2, (s_{10}, s_{20}), F_3 >$, where $F_3 = (F_1 \times S_1) \cup (S_1 \times F_2)$. Clearly, $T(\mathfrak{M}_3) = T(\mathfrak{M}_1) \cup T(\mathfrak{M}_2)$.

By Lemma 3 for $i = 1, 2$ we can determine 1-f.a.'s $\mathfrak{M}_i' = < S_i', \Sigma, M_i', s_{i0}', \{f_i\} >$, where for all $\sigma \in \Sigma$, $M_i'(f_i, \sigma) = \{f_i\}$, and such that $\mathfrak{M}_i$ and $\mathfrak{M}_i'$ are equivalent.

Define 1-f.a. $\mathfrak{M}_4 = < S_1' \times S_2', \Sigma, M_1' \times M_2', (s_{10}', s_{20}'), \{(f_1, f_2)\} >$. Clearly, $T(\mathfrak{M}_4) = T(\mathfrak{M}_1) \cap T(\mathfrak{M}_2)$. □

Theorem 11: 1'-f.a. are closed under union and intersection.

Proof: Given 1'-f.a.'s $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, F_1 >$, and $\mathfrak{M}_2 = < S_2, \Sigma, M_2, s_{20}, F_2 >$.

Define 1'-f.a. $\mathfrak{M}_3 = < S_1 \times S_2, \Sigma, M_1 \times M_2, (s_{10}, s_{20}), F_3 >$, where $F_3 = F_1 \times F_2$. Clearly, $T(\mathfrak{M}_3) = T(\mathfrak{M}_1) \cap T(\mathfrak{M}_2)$.

By Lemma 4 for i = 1, 2 we can determine 1'-f.a.'s $\mathfrak{M}_i' = < S_i', \Sigma, M_i', s_{i0}', F_i' >$, where $S_i' = F_i' \cup \{s_{Ti}\}$, $s_{i0}' \in F_i'$, and for all $\sigma \in \Sigma$, $M'(s_{Ti}, \sigma) = \{s_{Ti}\}$.

Define 1'-f.a. $\mathfrak{M}_4 = < S_1' \times S_2', \Sigma, M_1' \times M_2', (s_{10}', s_{20}'), F_4 >$, where $F_4 = (S_1' \times F_2') \cup (F_1' \times S_2')$. By Lemmas 1 and 2, for i = 1, 2 we have $(\forall v \in \Sigma^\omega)(Rn(\mathfrak{M}_i, v) \neq \emptyset)$. Hence, if there is an accepting $\mathfrak{M}_1$-run on v, or an accepting $\mathfrak{M}_2$-run on v, or both, then there is an accepting $\mathfrak{M}_4$-run on v. Therefore, $T(\mathfrak{M}_4) \supseteq T(\mathfrak{M}_1) \cup T(\mathfrak{M}_2)$. By the construction of $\mathfrak{M}_i'$ for all $v \in \Sigma^\omega$, $(\forall r \in Rn(\mathfrak{M}_i, v))(\forall t)(r(t) \in F_i'$ implies $r([t]) \subseteq F_i')$. Hence, there is an accepting $\mathfrak{M}_4$-run on v only if there is an accepting $\mathfrak{M}_1'$-run on v, or an accepting $\mathfrak{M}_2'$-run on v, or both. Therefore, $T(\mathfrak{M}_4) \subseteq T(\mathfrak{M}_1) \cup T(\mathfrak{M}_2)$. □

Theorem 12: 2-f.a. are closed under union and intersection.

Proof: Given 2-f.a.'s $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, F_1 >$ and $\mathfrak{M}_2 = < S_2, \Sigma, M_2, s_{20}, F_2 >$.

First we show closure under union. Define 2-f.a. $\mathfrak{M}_3 = < S_1 \times S_2, \Sigma, M_1 \times M_2, (s_{10}, s_{20}), F_3 >$, where $F_3 = (S_1 \times F_2) \cup (F_1 \times S_2)$. By Lemmas 1 and 2, for $i = 1, 2$, and all $v \in \Sigma^\omega$, $Rn(\mathfrak{M}_i, v) \neq \emptyset$. Hence, if there is an accepting $\mathfrak{M}_1$-run on v, or an accepting $\mathfrak{M}_2$-run on v, then there is an accepting $\mathfrak{M}_3$-run on v. Hence, $T(\mathfrak{M}_3) \supseteq T(\mathfrak{M}_1) \cup T(\mathfrak{M}_2)$. For all $v \in \Sigma^\omega$, and all $r_3 \in Rn(\mathfrak{M}_3, v)$, $p_1 r_3 \in Rn(\mathfrak{M}_1, v)$ and $p_2 r_3 \in Rn(\mathfrak{M}_2, v)$. Hence from the definition of F_3, if r_3 is an accepting $\mathfrak{M}_3$-run on v then either $p_1 r_3$ is an accepting $\mathfrak{M}_1$-run on v, or $p_2 r_3$ is an accepting $\mathfrak{M}_2$-run on v, or both. Hence $T(\mathfrak{M}_3) \subseteq T(\mathfrak{M}_1) \cup T(\mathfrak{M}_2)$.

We now show closure under intersection. Define 2-f.a. $\mathfrak{M}_4 = < S_4, \Sigma, M_4, (s_{10}, s_{20}, 1), F_4 >$, where $S_4 = S_1 \times S_2 \times \{1, 2\}$, for all $(s_1, s_2, i) \in S_4$, and all $\sigma \in \Sigma$, $M_4((s_1, s_2, i), \sigma) = \{(s_1', s_2', j) \in S_4 \mid s_1' \in M_1(s_1, \sigma) \ \& \ s_2' \in M_2(s_2, \sigma) \ \& \ (i = j \text{ iff } s_i \notin F_i)$, and $F_4 = S_1 \times F_2 \times \{2\}$.

Note that if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are deterministic then $\mathfrak{M}_4$ is deterministic.

Suppose $v \in T(\mathfrak{M}_1) \cap T(\mathfrak{M}_2)$. Hence, there exist $r_1 \in Rn(\mathfrak{M}_1, v)$ and $r_2 \in Rn(\mathfrak{M}_2, v)$ such that $In(r_1) \cap F_1 \neq \emptyset$ and $In(r_2) \cap F_2 \neq \emptyset$. From the definition of M_4, there exists $r_4 \in Rn(\mathfrak{M}_4, v)$ such that $p_1 r_4 = r_1$ and $p_2 r_4 = r_2$ (in fact, r_4 is unique). Suppose there exists a $\tau \in \mathbb{N}$ such that for all $t \geq \tau$, $r_4(t) \notin F_4$. But there exist t_1, t_2, t_2' such that $\tau < t_2 < t_1 < t_2'$, $r_1(t_1) \in F_1$, $r_2(t_2) \in F_2$, and $r_2(t_2') \in F_2$. Clearly from the definition of M_4, for some t, $t_2 \leq t \leq t_2'$, $r_4(t) \in F_4$. Hence, no such τ exists, and $In(r_4) \cap F_4 \neq \emptyset$. Hence, r_4 is an accepting $\mathfrak{M}_4$-run on v and $v \in T(\mathfrak{M}_4)$.

Suppose $v \in T(\mathfrak{M}_4)$. Then there exists $r_4 \in Rn(\mathfrak{M}_4, v)$ such that $In(r_4) \cap F_4 \neq \phi$. Hence, there exist $t_0 < t_1 < \ldots$, such that for all $i \in \mathbb{N}$, $r_4(t_i) \in F_4$. By the definition of M_4, $p_1 r_4 \in Rn(\mathfrak{M}_1, v)$ and $p_2 r_4 \in Rn(\mathfrak{M}_2, v)$. We have immediately from the definition of F_4, that for all $i \in \mathbb{N}$, $p_2 r_4(t_i) \in F_2$. Hence, $In(p_2 r_4) \cap F_2 \neq \phi$, and $v \in T(\mathfrak{M}_2)$. By the definition of M_4, there exist $t_0' < t_1' < \ldots$, such that $t_0' < t_0 < t_1' < t_1 \ldots$, and for all $i \in \mathbb{N}$, $p_1 r_4(t_i') \in F_1$. Hence, $In(p_1 r_4) \cap F_1 \neq \phi$, and $v \in T(\mathfrak{M}_1)$. Hence, $v \in T(\mathfrak{M}_1) \cap T(\mathfrak{M}_2)$.

Therefore, $T(\mathfrak{M}_4) = T(\mathfrak{M}_1) \cap T(\mathfrak{M}_2)$. $\square$

<u>Theorem 13</u>: 2'-f.a., 3-f.a., 4-f.a., 2'C-f.a., 3C-f.a., and 4C-f.a. are all closed under intersection and union.

<u>Proof</u>: Given 2'-f.a.'s (3-f.a.'s, 2'C-f.a.'s, 3C-f.a.'s) $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, \mathcal{F}_1 >$ and $\mathfrak{M}_2 = < S_2, \Sigma, M_2, s_{20}, \mathcal{F}_2 >$.

Define 2'-f.a. (3-f.a., 2'C-f.a., 3C-f.a., respectively) $\mathfrak{M}_3 = < S_1 \times S_2, \Sigma, M_1 \times M_2, (s_{10}, s_{20}), \mathcal{F}_3 >$, where $\mathcal{F}_3 = \{F \subseteq (S_1 \times S_2) \mid p_1 F \in \mathcal{F}_1 \;\&\; p_2 F \in \mathcal{F}_2\}$. Clearly, $T(\mathfrak{M}_3) = T(\mathfrak{M}_1) \cap T(\mathfrak{M}_2)$.

Define 2'-f.a. (3-f.a., 2'C-f.a., 3C-f.a., respectively) $\mathfrak{M}_4 = < S_1 \times S_2, \Sigma, M_1 \times M_2, (s_{10}, s_{20}), \mathcal{F}_4 >$, where $\mathcal{F}_4 = \{F \subseteq (S_1 \times S_2) \mid p_1 F \in \mathcal{F}_1$ or $p_2 F \in \mathcal{F}_2\}$. As in Theorems 10, 11, and 12 after an appeal to Lemmas 1 and 2, it is clear that $T(\mathfrak{M}_4) = T(\mathfrak{M}_1) \cup T(\mathfrak{M}_2)$.

Given 4-f.a.'s (4C-f.a.'s) $\mathfrak{M}_5 = < S_5, \Sigma, M_5, s_{50}, ((R_{5i}, G_{5i}))_{i \leq n} >$ and $\mathfrak{M}_6 = < S_6, \Sigma, M_6, s_{60}, ((R_{6j}, G_{6j}))_{j \leq m} >$.

First we show 4-f.a. and 4C-f.a. are closed under union. Define 4-f.a. (4C-f.a., respectively) $\mathfrak{M}_7 = < S_5 \times S_6, \Sigma, M_5 \times M_6, (s_{50}, s_{60}), \Omega_7 >$, where $\Omega_7 = ((R_{7k}, G_{7k}))_{k \leq n+m+1}$, and for all $0 \leq k \leq n$, $R_{7k} = R_{5k} \times S_6$ and $G_{7k} = G_{5k} \times S_6$, and for all $n+1 \leq k \leq n+m+1$, $R_{7k} = S_5 \times R_{6(k-n-1)}$ and $G_{7k} = S_5 \times G_{6(k-n-1)}$.

Suppose $v \in T(\mathfrak{M}_7)$. Then there exists $r_7 \in Rn(\mathfrak{M}_7, v)$ such that

1) for some $i \leq n$, $In(p_1 r_7) \cap R_{5i} = \phi$ and $In(p_1 r_7) \cap G_{5i} \neq \phi$, or

2) for some $j \leq m$, $In(p_2 r_7) \cap R_{6j} = \phi$ and $In(p_2 r_7) \cap G_{6j} \neq \phi$, or both.

Since $p_1 r_7 \in Rn(\mathfrak{M}_5, v)$ and $p_2 r_7 \in Rn(\mathfrak{M}_6, v)$, in case 1) $v \in T(\mathfrak{M}_5)$ and in case 2) $v \in T(\mathfrak{M}_6)$. Hence, $v \in T(\mathfrak{M}_5) \cup T(\mathfrak{M}_6)$.

Suppose $v \in T(\mathfrak{M}_5)$. Then there exists $r_5 \in Rn(\mathfrak{M}_5, v)$ such that for some $i \leq n$, $In(r_5) \cap R_{5i} = \phi$ and $In(r_5) \cap G_{5i} \neq \phi$. By Lemmas 1 and 2, $Rn(\mathfrak{M}_6, v) \neq \phi$, and hence, there exists $r_7 \in Rn(\mathfrak{M}_7, v)$ such that $p_1 r_7 = r_5$. Clearly, $In(r_7) \cap (R_{5i} \times S_6) = \phi$ and $In(r_7) \cap (G_{5i} \times S_6) \neq \phi$. Hence, r_7 is an accepting $\mathfrak{M}_7$-run on v, and $v \in T(\mathfrak{M}_7)$. Similarly, if $v \in T(\mathfrak{M}_6)$, then $v \in T(\mathfrak{M}_7)$.

Therefore, $T(\mathfrak{M}_7) = T(\mathfrak{M}_5) \cup T(\mathfrak{M}_6)$.

Finally, we show 4-f.a. and 4C-f.a. are closed under intersection. Our construction of $\mathfrak{M}_8$ such that $T(\mathfrak{M}_8) = T(\mathfrak{M}_5) \cap T(\mathfrak{M}_6)$ is somewhat complicated. However, it is merely an obvious extension of the construction used in Theorem 12 to show that 2-f.a. are closed under intersection, and we trust the reader will easily grasp the simple motivating notions behind our regrettably complex machinery. The state set of $\mathfrak{M}_8$ will be $S_8 = S_5 \times S_6 \times \beta$, where $\beta = \{B \mid B$ is an $(n+1)$ by $(m+1)$ matrix, $B = (b_{ij})_{i \leq n,\, j \leq m}$, each of whose entries is either 5 or 6, $b_{ij} \in \{5, 6\}\}$.

Define 4-f.a. (4C-f.a., respectively) $\mathfrak{M}_8 = < S_8, \Sigma, M_8, s_{80}, \Omega_8 >$, where S_8 is as above, for all $(s_5, s_6, B) \in S_8$, and all $\sigma \in \Sigma$, $M_8((s_5, s_6, B), \sigma) = \{(s_5', s_6', B') \in S_8 \mid s_5' \in M_5(s_5, \sigma) \& s_6' \in M_6(s_6, \sigma) \& (b_{ij}' = b_{ij}$ iff $s_{b_{ij}} \notin G_{5i}$ and $s_{b_{ij}} \notin G_{6j})\}$, $s_{80} = (s_{50}, s_{60}, B^0)$, where for all $i \leq n$, $j \leq m$, $b_{ij}^0 = 5$; and $\Omega_8 = ((R_{8ij}, G_{8ij}))_{i\leq n,\ j\leq m}$, where $R_{8ij} = R_{5i} \times S_6 \times \beta \cup S_5 \times R_{6j} \times \beta$, and $G_{8ij} = S_5 \times G_{6j} \times \{B \in \beta \mid b_{ij} = 6\}$.

Note that if $\mathfrak{M}_5$ and $\mathfrak{M}_6$ are deterministic then $\mathfrak{M}_8$ is deterministic.

Suppose $v \in T(\mathfrak{M}_8)$. Then there exists $r_8 \in Rn(\mathfrak{M}_8, v)$ such that for some $i \leq n$, and some $j \leq m$, $In(r_8) \cap R_{8ij} = \phi$ and $In(r_8) \cap G_{8ij} \neq \phi$. Clearly from the definitions of M_8 and Ω_8, we have $p_1r_8 \in Rn(\mathfrak{M}_5, v)$, $In(p_1r_8) \cap R_{5i} = \phi$, $p_2r_8 \in Rn(\mathfrak{M}_6, v)$, and $In(p_2r_8) \cap R_{6j} = \phi$. There exist $t_0 < t_1 < t_2 \ldots$, such that for all $i \in \mathbb{N}$, $r_8(t_i) \in G_{8ij}$. From the definition of Ω_8, for all $i \in \mathbb{N}$, $p_2r_8(t_i) \in G_{6j}$. From the definition of M_8, there exist $t_0' < t_1' < t_2' \ldots$, such that $t_0' < t_0 < t_1' < t_1 \ldots$, and for all $i \in \mathbb{N}$, $p_1r_8(t_i') \in G_{5i}$. Hence, $In(p_1r_8) \cap G_{5i} \neq \phi$ and $In(p_2r_8) \cap G_{6j} \neq \phi$. **Hence,** p_1r_8 is an accepting $\mathfrak{M}_5$-run on v and p_2r_8 is an accepting $\mathfrak{M}_6$-run on v. Therefore, $v \in T(\mathfrak{M}_5) \cap T(\mathfrak{M}_6)$.

Suppose $v \in T(\mathfrak{M}_5) \cap T(\mathfrak{M}_6)$. Then there exist $r_5 \in Rn(\mathfrak{M}_5, v)$ and $r_6 \in Rn(\mathfrak{M}_6, v)$ such that for some $i \leq n$, and some $j \leq m$, $In(r_5) \cap R_{5i} = \phi$, $In(r_5) \cap G_{5i} \neq \phi$, $In(r_6) \cap R_{6j} = \phi$, and $In(r_6) \cap G_{6j} \neq \phi$. Clearly, there exists a (unique, in fact) $r_8 \in Rn(\mathfrak{M}_8, v)$ such that $p_1r_8 = r_5$ and $p_2r_8 = r_6$. Clearly, $In(r_8) \cap R_{8ij} = \phi$.

Suppose there exists a $\tau \in \mathbb{N}$ such that for all $t \geq \tau$, $r_8(t) \notin G_{8ij}$. There exist t_5, t_6, t_6' such that $\tau < t_6 < t_5 < t_6'$, and $r_5(t_5) \in G_{5i}$, $r_6(t_6) \in G_{6j}$, and $r_6(t_6') \in G_{6j}$. But then from the definition of M_8, there must exist t such that $t_6 \leq t \leq t_6'$ and $r_8(t) \in G_{8ij}$; and this contradicts our assumption about τ. Thus no such τ exists and $In(r_1) \cap G_{8ij} \neq \phi$. Hence, r_8 is an accepting $\mathfrak{M}_8$-run on v.

Therefore, $T(\mathfrak{M}_8) = T(\mathfrak{M}_5) \cap T(\mathfrak{M}_6)$. □

For nondeterministic finite automata there is an obvious alternative construction of a "union machine" as follows.

Given tables $\mathfrak{M}_1' = < S_1, \Sigma, M_1, s_{10} >$ and $\mathfrak{M}_2' = < S_2, \Sigma, M_2, s_{20} >$. Define n-table $\mathfrak{M}_3' = < S_3, \Sigma, M_3, s_{30} >$, where $S_3 = S_1 \cup S_2 \{s_{30}\}$, $M_3 \mid (S_1 \cup S_2) = M_1 \cup M_2$, and for all $\sigma \in \Sigma$, $M_3(s_{30}, \sigma) = M_1(s_{10}, \sigma) \cup M_2(s_{20}, \sigma)$.

Clearly, for all $v \in \Sigma^\omega$, $Rn(\mathfrak{M}_3', v) = \{r_3 : \mathbb{N} \to S_3 \mid r_3(0) = s_{30}$ & (there exists $r_1 \in Rn(\mathfrak{M}_1', v)$ such that for all $t > 0$, $r_3(t) = r_1(t)$, or there exists $r_2 \in Rn(\mathfrak{M}_2', v)$ such that for all $t > 0$, $r_3(t) = r_2(t))\}$. Using this observation the correctness of each of the following constructions is immediate.

Given 1-f.a.'s (1'-f.a.'s, 2-f.a.'s, 1C-f.a.'s, 1'C-f.a., 2C-f.a.'s) $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, F_1 >$ and $\mathfrak{M}_2 = < S_2, \Sigma, M_2, s_{20}, F_2 >$, where $< S_1, \Sigma, M_1, s_{10} > = \mathfrak{M}_1'$ and $< S_2, \Sigma, M_2, s_{20} > = \mathfrak{M}_2'$. Define 1-n.f.a. (1'-n.f.a., 2-n.f.a., 1C-n.f.a., 1'C-n.f.a., 2C-n.f.a., respectively) $\mathfrak{M}_3 = < S_3, \Sigma, M_3, s_{30}, F_3 >$, where for $\mathfrak{M}_3$ a 1-n.f.a. or a 1C-n.f.a.

$$F_3 = \begin{cases} F_1 \cup F_2, & \text{if } s_{10} \notin F_1 \text{ and } s_{20} \notin F_2 \\ \{s_{30}\}, & \text{else,} \end{cases}$$

for $\mathfrak{M}_3$ a 1'-n.f.a. or a 1'C-n.f.a.

$$F_3 = \begin{cases} F_1 \cup F_2 \cup \{s_{30}\}, & \text{if } s_{10} \in F_1 \text{ and } s_{20} \in F_2 \\ \emptyset, & \text{else,} \end{cases}$$

and for $\mathfrak{M}_3$ a 2-n.f.a. or a 2C-n.f.a. $F_3 = F_1 \cup F_2$. $T(\mathfrak{M}_3) = T(\mathfrak{M}_1) \cup T(\mathfrak{M}_2)$.

Given 2'-f.a.'s (3-f.a.'s, 2'C-f.a.'s, 3C-f.a.'s) $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, \mathcal{F}_1 >$ and $\mathfrak{M}_2 = < S_2, \Sigma, M_2, s_{20}, \mathcal{F}_2 >$, where $< S_2, \Sigma, M_2, s_{20} > = \mathfrak{M}_2'$ and $< S_1, \Sigma, M_1, s_{10} > = \mathfrak{M}_1'$. Define 2'-n.f.a. (3-n.f.a., 2'C-n.f.a., 3C-n.f.a., respectively) $\mathfrak{M}_3 = < S_3, \Sigma, M_3, s_{30}, \mathcal{F}_3 >$, where $\mathcal{F}_3 = \mathcal{F}_1 \cup \mathcal{F}_2$. $T(\mathfrak{M}_3) = T(\mathfrak{M}_1) \cup T(\mathfrak{M}_2)$.

Given 4-f.a.'s (4C-f.a.'s) $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, ((R_{1i}, G_{1i}))_{i\leq n} >$, and $\mathfrak{M}_2 = < S_2, \Sigma, M_2, s_{20}, ((R_{2i}, G_{2i}))_{i\leq m} >$, where $< S_1, \Sigma, M_1, s_{10} > = \mathfrak{M}_1'$ and $< S_2, \Sigma, M_2, s_{20} > = \mathfrak{M}_2'$. Define 4-n.f.a. (4C-n.f.a., respectively) $\mathfrak{M}_3 = < S_3, \Sigma, M_3, s_{30}, \Omega >$, where $\Omega = ((R_{10}, G_{10}), \ldots, (R_{1n}, G_{1n}), (R_{20}, G_{20}), \ldots, (R_{2m}, G_{2m}))$. $T(\mathfrak{M}_3) = T(\mathfrak{M}_1) \cup T(\mathfrak{M}_2)$.

Theorem 14: 3-d.f.a. are closed under complementation.

Proof: Given 3-d.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, \mathcal{F} >$. Define 3-d.f.a. $\mathfrak{M}_1 = < S, \Sigma, M, s_0, P(S)-\mathcal{F} >$.

Because $\mathfrak{M}$ and $\mathfrak{M}_1$ are deterministic and have exactly the same run on each $v \in \Sigma^\omega$, clearly we have $(\exists r \in Rn(\mathfrak{M}, v))(In(r) \in \mathcal{F})$ iff not $(\exists r_1 \in Rn(\mathfrak{M}_1, v))(In(r_1) \in P(S)-\mathcal{F})$. Therefore, $T(\mathfrak{M}_1) = \Sigma^\omega - T(\mathfrak{M})$. □

SECTION VI COMPARISONS OF MODELS

In a natural intuitive sense j-f.a. are more powerful than i-f.a. (i-f.a. are incomparable in power to j-f.a.) when j-f.a. $\supset$ i-f.a. (when (i-f.a. $\not\supseteq$ j-f.a. & i-f.a. $\not\subseteq$ j-f.a.), respectively). In this intuitive sense, this section deals with the comparison of models.

Many of the results of the form i-f.a. $\subset$ j-f.a. are immediate corollaries of the failure of i-f.a., the weaker model, to be closed under complementation (projection), the closure of j-f.a., the stronger model, under complementation (projection), and a simple construction for obtaining an equivalent j-f.a. given an i-f.a. In fact, this is how we proceed in most of the following. However, we do not always proceed in the above manner. For example, we show 2-d.f.a. $\subset$ 2-n.f.a. by showing 2-n.f.a. $\equiv$ 3-n.f.a. and 2-d.f.a. $\subset$ 3-n.f.a. (2-d.f.a. $\subset$ 2-n.f.a., also, follows from 2-d.f.a. are not closed under projection, and 2-n.f.a. are closed under projection).

As is usual with negative results, the negative results in this section (for example, 2-d.f.a. are not closed complementation) are harder to prove than the positive results of the last section (for example, 2-d.f.a. are closed under union). Perhaps this is because rather than explaining, for example, how to put two 2-d.f.a.'s together into a new "union" machine, we must first find a task which no 2-d.f.a. can do, and then we must explain why no trick (no matter how brilliant) can devise a 2-d.f.a. which does the task. Hence, these negative results

<u>Lemma 9</u>: 2-n.f.a. $\subseteq$ 3-d.f.a., 2'-n.f.a. $\subseteq$ 3-d.f.a.

<u>Proof</u>: Given 2-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, F >$. Define 3-n.f.a. $\mathfrak{M}_1 = < S, \Sigma, M, s_0, \mathcal{F}_1 >$, where $\mathcal{F}_1 = \{F_1 \subseteq S \mid F_1 \cap F \neq \emptyset\}$. Clearly, $T(\mathfrak{M}_1) = T(\mathfrak{M})$. Hence, 2-n.f.a. $\subseteq$ 3-n.f.a., and by Theorem 5, 2-n.f.a. $\subseteq$ 3-d.f.a.

Given 2'-d.f.a. $\mathfrak{M}_2 = < S_2, \Sigma, M_2, s_{20}, \mathcal{F}_2 >$. Define 3-d.f.a. $\mathfrak{M}_3 = < S_2, \Sigma, M_2, s_{20}, \mathcal{F}_3 >$, where $\mathcal{F}_3 = \{F_3 \subseteq S_2 \mid \text{for some } F \in \mathcal{F}_2, F_3 \subseteq F\}$. Clearly, $T(\mathfrak{M}_3) = T(\mathfrak{M}_2)$. Hence, 2'-d.f.a. $\subseteq$ 3-d.f.a. and by Theorem 3, 2'-n.f.a. $\subseteq$ 3-d.f.a. □

<u>Theorem 15</u>: $A \subseteq \Sigma^\omega$ is 1-d.f.a. definable iff $A^c = \Sigma^\omega - A$ is 1'-d.f.a. definable.

<u>Proof</u>: Given 1-d.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, F >$. Define 1'-d.f.a. $\mathfrak{M}_1 = < S, \Sigma, M, s_0, S\text{-}F >$. Given 1'-d.f.a. $\mathfrak{M}_2 = < S_2, \Sigma, M_2, s_{20}, F_2 >$. Define 1-d.f.a. $\mathfrak{M}_3 = < S_2, \Sigma, M_2, s_{20}, S_2\text{-}F_2 >$.

For all $v \in \Sigma^\omega$, $\mathfrak{M}$ and $\mathfrak{M}_1$ have precisely the same unique run on v. Hence, $(\exists r \in Rn(\mathfrak{M}, v))(r(/\!N) \cap F \neq \emptyset)$ iff not $(\exists r \in Rn(\mathfrak{M}_1, v))(r(/\!N) \subseteq S\text{-}F)$. Therefore, $T(\mathfrak{M}_1) = \Sigma^\omega - T(\mathfrak{M})$.

For all $v \in \Sigma^\omega$, $\mathfrak{M}_2$ and $\mathfrak{M}_3$ have precisely the same unique run on v. Hence, $(\exists r \in Rn(\mathfrak{M}_2, v))(r(/\!N) \subseteq F_2)$ iff not $(\exists r \in Rn(\mathfrak{M}_3, v))(r(/\!N) \cap S_2\text{-}F_2 \neq \emptyset)$. Therefore, $T(\mathfrak{M}_3) = \Sigma^\omega - T(\mathfrak{M}_2)$. □

<u>Lemma 10</u>: $A_1 = 0^\omega$ is not 1-n.f.a. definable.

<u>Proof</u>: Suppose $A_1 = 0^\omega$ is defined by 1-n.f.a. $\mathfrak{M} = < S, \{0, 1\}, M, s_0, F >$. Then there exists $r \in Rn(\mathfrak{M}, 0^\omega)$ and $\tau \in \mathbb{N}$ such that $r(\tau) \in F$.

Consider $0^\tau \cdot 1^\omega$. $r \mid [\tau+1]$ is compatible with $\mathfrak{M}$ and $0^\tau \cdot 1^\omega$. Hence, by Lemma 1, there exists $r' \in Rn(\mathfrak{M}, 0^\tau \cdot 1^\omega)$ such that $r' \mid [\tau+1] = r \mid [\tau+1]$. But $r'(\tau) = r(\tau) \in F$, and hence, $0^\tau \cdot 1^\omega \in T(\mathfrak{M})$. Contrary to assumption $\mathfrak{M}$ does not define 0^ω. Therefore, 0^ω is not 1-n.f.a. definable. □

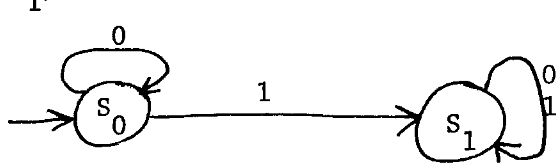
<u>Lemma 11</u>: $A_1^c = \{v \in \{0, 1\}^\omega \mid 1 \in v(\mathbb{N})\} = 0^* \cdot 1 \cdot \{0, 1\}^\omega$ is not 1'-n.f.a. definable.

<u>Proof</u>: Suppose A_1^c is defined by 1'-n.f.a. $\mathfrak{M} = < S, \{0, 1\}, M, s_0, F >$, where $c(S) = n$.

$0^n \cdot 1^\omega \in A_1^c$. Hence, there exists $r \in Rn(\mathfrak{M}, 0^n \cdot 1^\omega)$ such that $r(\mathbb{N}) \subseteq F$. $c(\{0, \ldots, n\}) > n$, hence for some $t_1 < t_2 \leq n$, $r(t_1) = r(t_2)$. Hence, there exists $r' \in Rn(\mathfrak{M}, 0^\omega)$ such that $r' = r(0) \cdot r(1) \ldots\ldots r(t_1 - 1) \cdot (r(t_1) \ldots r(t_2 - 1))^\omega$. Clearly $r'(\mathbb{N}) \subseteq F$, and hence, $0^\omega \in T(\mathfrak{M})$. Therefore, contrary to assumption A_1^c is not 1'-nf.a. definable. □

<u>Theorem 16</u>: 1-f.a. and 1'-f.a. are not closed under complementation, 1-f.a. and 1'-f.a. are incomparable, 1-n.f.a. $\subset$ 2-d.f.a., 1-n.f.a. $\subset$ 2'-d.f.a., 1-n.f.a. $\subset$ 3-d.f.a., 1'-n.f.a. $\subset$ 2-d.f.a., 1'-n.f.a. $\subset$ 2'-d.f.a., 1'-n.f.a. $\subset$ 3-d.f.a.

<u>Proof</u>: $A_1^c = \{v \in \{0, 1\}^\omega \mid 1 \in v(\mathbb{N})\}$ is defined by 1-d.f.a. $\mathfrak{M} = < \{s_0, s_1\}, \{0, 1\}, M, s_0, \{s_1\} >$, where M is given by:



.

Hence by Lemma 10, 1-f.a. are not closed under complementation.

$A_1 = 0^\omega$ is defined by 1'-d.f.a. $\mathfrak{M}_1 = < \{s_0, s_1\}, \{0, 1\}, M, s_0,$ $\{s_0\} >$, where M is as in the diagram above. Hence, by Lemma 11, 1'-f.a. are not closed under complementation.

The remaining parts of the theorem are immediate from the above, and Lemmas 7, 8, 10, and 11. □

It is interesting to note that $A_2 = 1^* 0^\omega$ is neither 1-n.f.a. definable nor 1'-n.f.a. definable; but A_2 is both 2-d.f.a. definable and 2'-d.f.a. definable.

<u>Theorem 17</u>: $A \subseteq \Sigma^\omega$ is 2-d.f.a. definable iff Σ^ω-A is 2'-d.f.a. definable.

<u>Proof</u>: Given 2-d.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, F >$. Define 2'-d.f.a. $\mathfrak{M}_1 =$ $< S, \Sigma, M, s_0, \{S-F\} >$. For all $v \in \Sigma^\omega$, $\mathfrak{M}$ and $\mathfrak{M}_1$ have precisely the same unique run on v. Hence, $(\exists r \in Rn(\mathfrak{M}, v))(In(r) \cap F \neq \emptyset)$ iff not $(\exists r \in Rn(\mathfrak{M}_1, v))(In(r) \subseteq (S-F))$. Therefore, $T(\mathfrak{M}_1) = (T(\mathfrak{M}))^c$.

Given 2'-d.f.a. on Σ^ω $\mathfrak{M}_2$, possibly with many designated subsets, we can find (by Theorem 21) a 2'-d.f.a. $\mathfrak{M}_3$ equivalent to $\mathfrak{M}_2$ and such that $\mathfrak{M}_3 = < S_3, \Sigma, M_3, s_{30}, \{F_3\} >$ (note that $\mathfrak{M}_3$ has only one designated subset). Define 2-d.f.a. $\mathfrak{M}_4 = < S_3, \Sigma, M_3, s_{30}, S_3-F_3 >$. For all $v \in$ Σ^ω, $\mathfrak{M}_3$ and $\mathfrak{M}_4$ have precisely the same unique run on v. Hence, $(\exists r \in$ $Rn(\mathfrak{M}_3, v))(In(r) \subseteq F_3)$ iff not $(\exists r \in Rn(\mathfrak{M}_4, v))(In(r) \cap (S_3-F_3) \neq \emptyset)$. Therefore, $T(\mathfrak{M}_4) = (T(\mathfrak{M}_3))^c$ □

The reader may be tempted to say that closure of the family of 2-d.f.a. definable sets under projection is a corollary of Theorem 17, reasoning as follows.

If set A is 2-d.f.a. definable, then set A^c is 2'-d.f.a. definable. Hence, $p_1(A^c)$ is 2'-d.f.a. definable (by Theorems 3 and 8). Hence, $(p_1(A^c))^c$ is 2-d.f.a. definable. But, in general, $(p_1(A^c))^c \neq p_1((A^c)^c) = p_1A$, because complementation and projection on regular events and on ω-regular events, in general, and on 2-d.f.a. definable sets, in particular, do not commute.

We will see that the family of 2-d.f.a. definable sets is, in fact, not closed under projection (Theorem 19).

<u>Lemma 12</u>: $A_3 = \{v \in \{0, 1\}^\omega \mid 1 \notin In(v)\} = \{0, 1\}^* \cdot 0^\omega$ is not 2-d.f.a. definable.

<u>Proof</u>: Suppose A_3 is defined by 2-d.f.a. $\mathfrak{M} = < S, \{0, 1\}, M, s_0, F >$, where $c(F) = k$.

Let $v_0 = 0^\omega$. Since $v_0 \in A_3$, $v_0 \in T(\mathfrak{M})$. Therefore, for the unique $r_0 \in Rn(\mathfrak{M}, v_0)$, we have $In(r_0) \cap F \neq \emptyset$. Let τ_0 be the least $t \in \mathbb{N}$ such that $r_0(t) \in F$.

For $i < \omega$, define v_{i+1}, r_{i+1}, τ_{i+1} inductively as follows. Let $v_{i+1} = v_i(0) \cdot v_i(1) \ldots v_i(\tau_i - 1) \cdot 1 \cdot 0^\omega$. Let r_{i+1} be the unique $\mathfrak{M}$-run on v_{i+1}. Since $v_{i+1} \in A_3$, $In(r_{i+1}) \cap F \neq \emptyset$. Therefore, we can let τ_{i+1} be the least $t > \tau_i$ such that $r_{i+1}(t) \in F$.

Using Lemma 1 we can show by induction that for all $i \le k+1$, we have $r_{k+1} \mid [\tau_i+1] = r_i \mid [\tau_i+1]$. (Note that Lemma 1 is not true for 2-n.f.a.'s. This is an essential use of $\mathfrak{M}$'s determinism.)

Hence, for all $i \le k+1$, we have $r_{k+1}(\tau_i) = r_i(\tau_i) \in F$. Since $c(F) = k$, there exist h, j such that $h < j \le k+1$ and $r_{k+1}(\tau_h) = r_{k+1}(\tau_j)$. We have the picture:

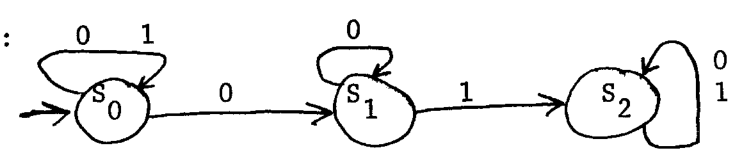
time:	0	1	τ_1-1	τ_1
v_{k+1}	0	0	0	1 ....
r_{k+1}:	$r_{k+1}(0)$	$r_{k+1}(1)$	$r_{k+1}(\tau_1-1)$	$r_{k+1}(\tau_1) \in F$ ...

time:	τ_i-1	τ_i		τ_{k+1}
v_{k+1}:	0	1	0	1 0^ω
r_{k+1}:	$r_{k+1}(\tau_i-1)$	$r_{k+1}(\tau_i) \in F$		$r_{k+1}(\tau_{k+1}) \in F$

Let $v = v_{k+1}(0) \cdots v_{k+1}(\tau_h-1)(v_{k+1}(\tau_h) \cdots v_{k+1}(\tau_j-1))^\omega$. Let $r = r_{k+1}(0) \cdots r_{k+1}(\tau_h-1)((r_{k+1}(\tau_h) \cdots r_{k+1}(\tau_j-1))^\omega$. Clearly, $r \in Rn(\mathfrak{M}, v)$ and $r(\tau_h) = r_{k+1}(\tau_h) \in F$. Hence, $In(r) \cap F \neq \emptyset$, and r is an accepting $\mathfrak{M}$-run on v. But $v(\tau_h) = v_{k+1}(\tau_h) = 1$, hence, $1 \in In(v)$, and $v \notin A_3$.

Therefore, contrary to our assumption, A_3 is not 2-d.f.a. definable. $\square$

As we noted the preceding proof makes essentail use of the determinism of machine $\mathfrak{M}$. In fact, Lemma 12 is not true for 2-n.f.a.'s. A_3 is defined by the 2-n.f.a. $\mathfrak{M} = < \{s_0, s_1, s_2\}, \{0, 1\}, M, s_0, \{s_1\} >$, where M is given by:



Lemma 13: $A_3^c = \{v \in \{0, 1\}^\omega \mid 1 \in In(v)\} = (0^* \cdot 1)^\omega$ is not 2'-n.f.a. definable.

Proof: Suppose A_3^c is defined by 2'-n.f.a. $\mathfrak{M} = \langle S, \{0, 1\}, M, s_0, \mathcal{F} \rangle$, where $c(S) = n$.

$v = (0^n \cdot 1)^\omega \in A_3$, and hence, $v \in T(\mathfrak{M})$. Therefore, there exists $r \in Rn(\mathfrak{M}, v)$ such that for some $F \in \mathcal{F}$, $In(r) \subseteq F$. Then there exists $\tau \in \mathbb{N}$ such that for all $t \geq \tau$, we have $r(t) \in F$. Choose any $k > \tau$ such that $v(k) = 1$. Then $v(k+1) \cdot v(k+2) \ldots v(k+n+1) = 0^n$, and for all t, $k < t \leq k+n+2$, we have $r(t) \in F$. Now $F \subseteq S$, and hence, $c(F) \leq n$ and there must exist t_1, t_2 such that $k < t_1 < t_2 \leq k+n+2$, and $r(t_1) = r(t_2)$.

We have the picture:

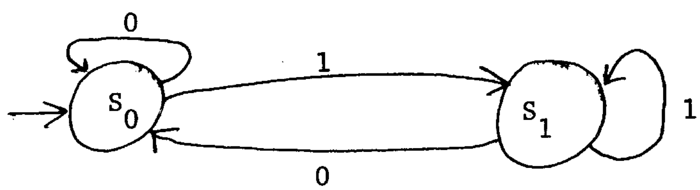
time:	k-1	k	k+1	....	k+n+1	k+n+2
v:	0	1	0	....	0	1
r:		r(k)	r(k+1)	....		r(k+n+2)

$k < t_1 < t_2 \leq k+n+2$

Let $v_1 = v(0) \ldots v(t_1-1) \cdot (v(t_1) \ldots v(t_2-1))^\omega$. Let $r_1 = r(0) \ldots r(t_1-1) \cdot (r(t_1) \ldots r(t_2-1))^\omega$. Clearly, $r_1 \in Rn(\mathfrak{M}, v_1)$ and $In(r_1) \subseteq In(r) \subseteq F$. Hence, r_1 is an accepting $\mathfrak{M}$-run on v_1. But $v_1 \in A_3^c$, and hence, contrary to our assumption, A_3^c is not 2'-n.f.a. definable. □

Theorem 18: 2-d.f.a. and 2'-f.a. are not closed under complementation, 2-d.f.a. and 2'-f.a. are incomparable, 2-d.f.a. $\subset$ 3-d.f.a., 2-d.f.a. $\subset$ 2-n.f.a., 2'-n.f.a. $\subset$ 3-d.f.a., and 2'-n.f.a. $\subset$ 2-n.f.a.

Proof: $A_3^c = (0^* \cdot 1)^\omega$ is defined by 2-d.f.a. $\mathfrak{M} = \langle \{s_0, s_1\}, \{0, 1\}, M, s_0, \{s_1\} \rangle$, where M is given by:



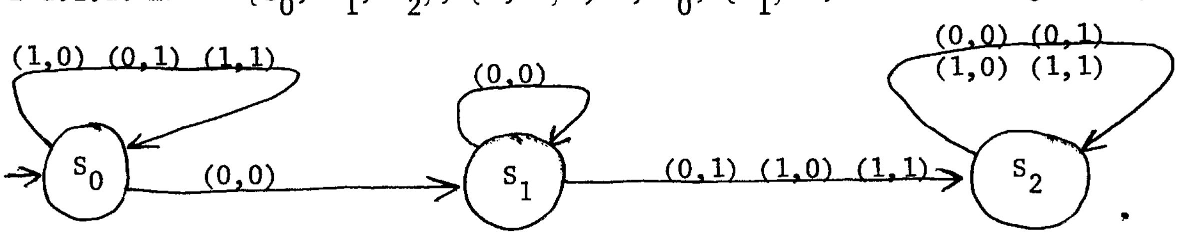
Hence by Lemma 12, 2-d.f.a. are not closed under complementation.

$A_3 = \{0, 1\}^* \cdot 0^\omega$ is defined by 2'-d.f.a. $\mathfrak{M}_1 = \langle \{s_0, s_1\}, \{0, 1\}, M, s_0, \{\{s_0\}\} \rangle$, where M is given by the state transition diagram above. Hence, by Lemma 13, 2'-f.a. are not closed under complementation.

The remaining parts of the theorem are immediate from the above, Lemmas 9, 12, and 13, and Theorems 4 and 5. □

Theorem 19: 2-d.f.a. are not closed under projection.

Proof: Let $A_4 = \{(0, 1), (1, 0), (1, 1)\}^* \cdot (0, 0)^\omega$. A_4 is defined by 2-d.f.a. $\mathfrak{M} = \langle \{s_0, s_1, s_2\}, \{0, 1\}^2, M, s_0, \{s_1\} \rangle$, where M is given by:



$p_1(A_4) = \{0, 1\}^* \cdot 0^\omega = A_3$. By Lemma 12, A_3 is not 2-d.f.a. definable, and hence, 2-d.f.a. are not closed under projection. □

Coincidentally, $p_1(A_4) = p_2(A_4)$. This is not essential.

Remark 1: Let $A_5 = \{v \in \{0, 1\}^\omega \mid v(0) = 1 \to 1 \notin In(v)$ and $v(0) = 0 \to 1 \in In(v)\} = 1 \cdot \{0, 1\}^* \cdot 0^\omega \cup 0 \cdot (0^* 1)^\omega$. A_5 is neither 2-d.f.a. definable nor 2'-n.f.a. definable; but A_5 is 3-d.f.a. definable.

Proof of Remark 1: Suppose A_5 is defined by 2-d.f.a. $\mathfrak{M} = \langle S, \{0, 1\}, M, s_0, F \rangle$. Define 2-d.f.a. $\mathfrak{M}_1 = \langle S_1, \{0, 1\}, M_1, s_{10}, F_1 \rangle$, where $S_1 = S \cup \{s_{10}\}$, for all $s \in S$, and all $\sigma \in \{0, 1\}$, $M_1(s, \sigma) = M(s, \sigma)$, $M_1(s_{10}, 0) = \{s_{10}\}$, $M_1(s_{10}, 1) = M(s_0, 1)$, and $F_1 = F \cup \{s_{10}\}$.

Let v_1 be any $\{0, 1\}^\omega$-sequence.

Case 1: $1 \in v_1(\mathbb{N})$.

Let τ be the least t such that $v_1(t) = 1$. Let $v = v_1(\tau) \cdot v_1(\tau+1) \cdot v_1(\tau+2) \ldots v_1(\tau+n) \ldots$. Let r be the unique $\mathfrak{M}$-run on v.

Clearly, for the unique $\mathfrak{M}_1$-run r_1 on v_1 we have for all $t \le \tau$, $r_1(t) = s_{10}$, and for all $k > 0$, $r_1(\tau+k) = r(k)$. Hence, $v_1 \in T(\mathfrak{M}_1)$ iff $v \in T(\mathfrak{M})$ iff $1 \notin In(v)$ iff $1 \notin In(v_1)$.

Case 2: $1 \notin v_1(\mathbb{N})$.

Then $v_1 = 0^\omega$ and the unique $r_1 \in Rn(\mathfrak{M}_1, 0^\omega)$ is $r_1 = s_{10}^\omega$. We have $s_{10} \in F_1$ and hence, $v_1 \in T(\mathfrak{M}_1)$. Clearly, $1 \notin In(v_1)$. This completes Case 2.

Therefore, $T(\mathfrak{M}_1) = \{v \in \{0, 1\}^\omega \mid 1 \notin In(v)\} = A_3$. But by Lemma 12, A_3 is not 2-d.f.a. definable and we have a contradiction. Hence, contrary to our assumption, A_5 is not 2-d.f.a. definable.

Suppose A_5 is defined by 2'-n.f.a. $\mathfrak{M}_2 = < S_2, \{0, 1\}, M_2, s_{20}, \mathcal{F}_2 >$. Define 2'-n.f.a. $\mathfrak{M}_3 = < S_3, \{0, 1\}, M_3, s_{30}, \mathcal{F}_3 >$, where $S_3 = S_2 \cup \{s_{30}\}$, for all $s \in S_2$, and all $\sigma \in \{0, 1\}$, $M_3(s, \sigma) = M_2(s, \sigma)$, $M_3(s_{30}, 1) = \{s_{30}\}$, $M_3(s_{30}, 0) = M_2(s_{20}, 0)$, and $\mathcal{F}_3 = \mathcal{F}_2 \cup \{\{s_{30}\}\}$.

Let v_3 be any $\{0, 1\}^{\omega}$-sequence.

<u>Case 1</u>: $0 \in v_3(\mathbb{N})$.

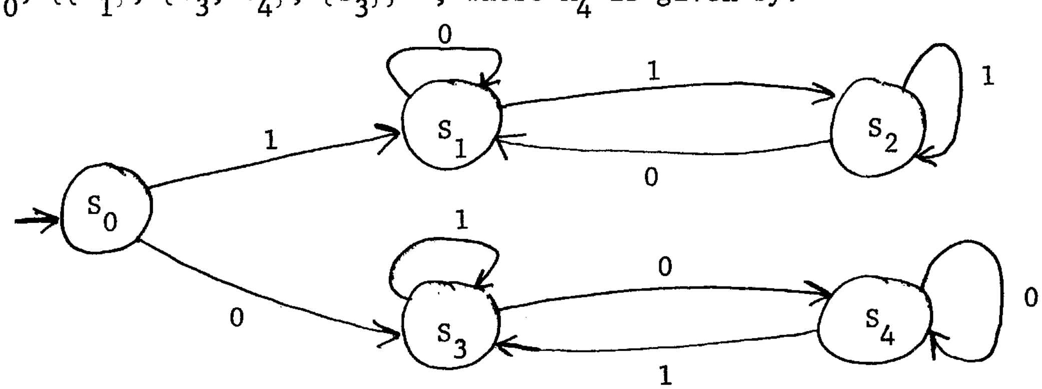
Let τ be the least t such that $v_3(t) = 0$. Let $v_2 = v_3(\tau)\cdot v_3(\tau+1)\cdot v_3(\tau+2) \ldots v_3(\tau+n) \ldots$. Clearly, for each $r_3 \in Rn(\mathfrak{M}_3, v_3)$ there exists an $r_2 \in Rn(\mathfrak{M}_2, v_2)$ such that for all $k > 0$, $r_3(\tau+k) = r_2(k)$; and for each $r_2 \in Rn(\mathfrak{M}_2, v_2)$ there exists an $r_3 \in Rn(\mathfrak{M}_3, v_3)$ such that for all $k > 0$, $r_2(k) = r_3(\tau+k)$. Hence, there exists $r_3 \in Rn(\mathfrak{M}_3, v_3)$ such that $In(r_3) \subseteq F \in \mathcal{F}_3$ iff there exists $r_2 \in Rn(\mathfrak{M}_2, v_2)$ such that $In(r_2) \subseteq F \in \mathcal{F}$. Hence, $v_3 \in T(\mathfrak{M}_3)$ iff $v_2 \in T(\mathfrak{M}_2)$ iff $1 \in In(v_2)$ iff $1 \in In(v_3)$.

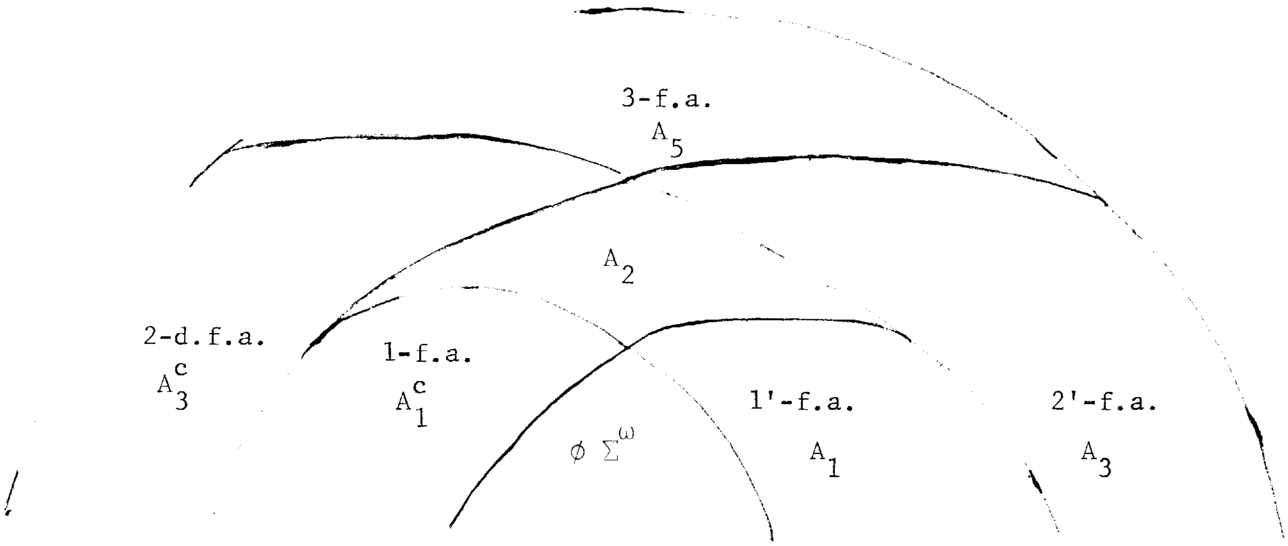
<u>Case 2</u>: $0 \notin v_3(\mathbb{N})$.

Then $v_3 = 1^{\omega}$, and the unique $\mathfrak{M}_3$-run on v_3 is $r_3 = s_{30}^{\omega}$. Since $\{s_{30}\} \in \mathcal{F}_3$, $v_3 \in T(\mathfrak{M}_3)$. This completes Case 2.

Therefore, $T(\mathfrak{M}_3) = \{v \in \{0, 1\}^{\omega} \mid 1 \in In(v)\} = A_3^c$. But by Lemma 13 A_3^c is not 2'-n.f.a. definable, and we have a contradiction. Hence, contrary to our assumption, A_5 is not 2'-n.f.a. definable.

A_5 is defined by 3-d.f.a. $\mathfrak{M}_4 = < \{s_0, s_1, s_2, s_3, s_4\}, \{0, 1\}, M_4, s_0, \{\{s_1\}, \{s_3, s_4\}, \{s_3\}\} >$, where M_4 is given by:





$$A_1 = 0^\omega$$

$$A_1^c = 0^* \cdot 1 \cdot \{0, 1\}^\omega = \{v \in \{0, 1\}^\omega \mid 1 \in v(\mathbb{N})\}$$

$$A_2 = 1^* \cdot 0^\omega$$

$$A_3 = \{0, 1\}^* \cdot 0^\omega = \{v \in \{0, 1\}^\omega \mid 1 \notin In(v)\}$$

$$A_3^c = (0^* \cdot 1)^\omega = \{v \in \{0, 1\}^\omega \mid 1 \in In(v)\}$$

$$A_5 = 1 \cdot \{0, 1\}^* \cdot 0^\omega \cup 0 \cdot (0^* \cdot 1)^\omega = \{v \in \{0, 1\}^\omega \mid (v(0) = 1 \rightarrow 1 \notin In(v)) \;\&\; (v(0) = 0 \rightarrow 1 \in In(v))\}$$

FIGURE 3

SECTION VIII RESTRICTING THE ω-AUTOMATA MODELS

Sometimes the most natural, general definition of an automaton model may be restricted so as to yield a model which is easier to handle for some proofs, and which is still as powerful as the unrestricted, original model (i.e., the same sets of Σ^{ω}-sequences are definable using either model). We have already seen two examples of this in Lemmas 3 and 4, which show that certain convenient restrictions may be placed on the initial state and the state transition functions of 1-f.a.'s and 1'-f.a.'s. In fact, Lemma 3 yields the following.

Theorem 20: Given a 1-n.f.a. (1-d.f.a.) we can determine an equivalent 1-n.f.a. (1-d.f.a., respectively) with a single designated state.

Proof: Immediate from Lemma 3. □

If we didn't restrict state transition functions to be mappings $M: S \times \Sigma \to P(S) - \{\emptyset\}$, then Theorem 20 would hold for 1-d.f.a.'s, but not for 1-n.f.a.'s.

Theorem 20 does not hold for 1'-f.a.'s for it is trivial to show that $A_8 = (01)^{\omega}$ is 1'-d.f.a. definable, but that A_8 is not definable by any 1'-n.f.a. with only one designated state.

It is, also, quite easy to show that A_8 is both 2'-d.f.a. and 3-d.f.a. definable, but that A_8 is not definable by any 2'-n.f.a. or 3-n.f.a. all of whose designated subsets are singleton sets.

It is easy to show that $A_7 = 0^\omega \cup 1^\omega$ is both 2-d.f.a. and 3-d.f.a. definable, but that A_7 is not definable by any 2-n.f.a. (3-n.f.a.) with only one designated state (subset, respectively).

Theorem 21: Given a 2'-n.f.a. (2'-d.f.a.) we can determine an equivalent 2'-n.f.a. (2'-d.f.a., respectively) which has only one designated subset.

Proof: Given 2'-f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, \{F_1, \ldots, F_k\} >$. For each $1 \le i \le k$, define 2'-f.a. $\mathfrak{M}_i = < S, \Sigma, M, s_0, \{F_i\} >$. Clearly, $T(\mathfrak{M}) = T(\mathfrak{M}_1) \cup \ldots \cup T(\mathfrak{M}_k)$. Hence, the following suffices.

Given 2'-f.a.'s $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, \{F_1\} >$ and $\mathfrak{M}_2 = < S_2, \Sigma, M_2, s_0, \{F_2\} >$. Define 2'-n.f.a. $\mathfrak{M}_3 = < S_1 \cup S_2 \cup \{s_{30}\}, \Sigma, M_3, s_{30}, \{F_1 \cup F_2\} >$, where for all $\sigma \in \Sigma$, $M_3(s_{30}, \sigma) = M_1(s_{10}, \sigma) \cup M_2(s_{20}, \sigma)$, and $M_3 \mid S_1 \cup S_2 = M_1 \cup M_2$. Note that this is the same construction presented immediately after the proof of Theorem 13.

Clearly, for all $v \in \Sigma^\omega$, $Rn(\mathfrak{M}_3, v) = \{r_3 : \mathbb{N} \to S_3 \mid r_3(0) = s_{30}$ & (there exists $r_2 \in Rn(\mathfrak{M}_2, v)$ such that for all $t > 0$, $r_3(t) = r_2(t)$, or there exists $r_1 \in Rn(\mathfrak{M}_1, v)$ such that for all $t > 0$, $r_1(t) = r_3(t))\}$. Hence, there exists $r_3 \in Rn(\mathfrak{M}_3, v)$ such that $In(r_1) \subseteq \{F_1 \cup F_2\}$ iff there exists $r_1 \in Rn(\mathfrak{M}_1, v)$ such that $In(r_1) \subseteq F_1$, or there exists $r_2 \in Rn(\mathfrak{M}_2, v)$ such that $In(r_2) \subseteq F_2$. Therefore, $T(\mathfrak{M}_3) = T(\mathfrak{M}_1) \cup T(\mathfrak{M}_2)$.

$\mathfrak{M}_3$ has only one designated subset. Hence, this completes the proof for 2'-n.f.a.'s.

Using the construction in the proof of Theorem 3, from 2'-n.f.a. $\mathfrak{M}_3$ we can construct an equivalent 2'-d.f.a. $\mathfrak{M}_4$ which has only one designated subset. □

The direct construction of a 2'-d.f.a. with only one designated subset from a 2'-d.f.a. with two designated subsets is complicated. In the preceding proof we avoided this complicated construction by the judicious use of previous, simpler constructions.

The more complicated the model is the more ways there are to restrict it. Indeed, the 4-f.a. model may be restricted in several useful ways. For example, the following three remarks are immediate from the definition of 4-accepting.

Remark 2: 4-f.a. (4C-f.a.) $\mathfrak{M} = < S, \Sigma, M, s_0, ((R_i, G_i))_{i \leq n} >$ is equivalent to 4-f.a. (4C-f.a., respectively) $\mathfrak{M}_1 = < S, \Sigma, M, s_0, ((R_i, (G_i - R_i)))_{i \leq n} >$.

Remark 3: Given 4-f.a. (4C-f.a.) $\mathfrak{M} = < S, \Sigma, M, s_0, ((R_i, G_i))_{i \leq n} >$. For each $i \leq n$, define 4-f.a. (4C-f.a., respectively) $\mathfrak{M}_i = < S, \Sigma, M, s_0, ((R_i, G_i)) >$. Clearly, $T(\mathfrak{M}) = T(\mathfrak{M}_0) \cup \ldots \cup T(\mathfrak{M}_n)$.

Remark 4: Given 4-f.a. (4C-f.a.) $\mathfrak{M} = < S, \Sigma, M, s_0, ((R, G)) >$, where $G = \{s_1, \ldots, s_k\}$. Define 4-f.a. (4C-f.a., respectively) $\mathfrak{M}_1 = < S, \Sigma, M, s_0, ((R, \{s_i\}))_{1 \leq i \leq k} >$. Clearly, $T(\mathfrak{M}) = T(\mathfrak{M}_1)$.

Theorem 22: Given any 4-n.f.a. we can determine an equivalent 4-n.f.a. with subset pairs $\Omega = ((\emptyset, G))$.

Proof: Given a 4-n.f.a. $\mathfrak{M}$ on Σ^ω by Theorem 6 (4-f.a. ≡ 3-f.a.), and Theorem 4 (2-n.f.a. ≡ 3-n.f.a.), we can determine an equivalent 2-n.f.a. $\mathfrak{M}_1 = < S_1, \Sigma, M_1, s_{10}, F_1 >$. Then 4-n.f.a. $\mathfrak{M}_2 = < S_1, \Sigma, M_1, s_{10}, ((\emptyset, F_1)) >$ is equivalent to $\mathfrak{M}_1$ and hence to $\mathfrak{M}$. □

Given a 4-n.f.a. we can not in general find an equivalent 4-n.f.a. with subset pairs $\Omega = ((\emptyset, \{f\}))$. For example, it is easily shown that the set $A_7 = 0^\omega \cup 1^\omega$ is not defined by any 4-n.f.a. with subset pairs $\Omega = ((\emptyset, \{f\}))$.

Given a 4-d.f.a. we can not in general find an equivalent 4-d.f.a. with subset pairs $\Omega = ((\emptyset, G))$. In fact, the set $A_3 = \{v \in \{0, 1\}^\omega | 1 \notin In(v)\}$ is 4-d.f.a. definable, but A_3 is not defined by any 4-d.f.a. with subset pairs $\Omega = ((\emptyset, G))$, for if it were, then we would immediately have a 2-d.f.a. defining A_3, and this would contradict Lemma 12. Similarly, we see that Theorem 22 does not hold for 4C-n.f.a.'s, for if A_3 were defined by some 4C-n.f.a. with pairs $\Omega = ((\emptyset, G))$, then we would immediately have a 2C-n.f.a. defining A_3, but A_3 is not 2C-n.f.a. definable (by Lemma 12 and Theorem 7).

SECTION IX COMMENTS ON THE C-RUN MODELS

All of the following are quite easily shown directly by construction:

1) 1C-, 1'C-, 2'C-, and 3C-f.a. are closed under union and intersection,

2) 2C-f.a. are closed under union,

3) 1C-n.f.a. are closed under projection,

4) 3C-d.f.a. are closed under complementation,

5) 1C-f.a. $\subseteq$ 2C-f.a., 1C-f.a. $\subseteq$ 2'C-f.a., 1C-f.a. $\subseteq$ 3C-f.a.,

6) 1'C-f.a. $\subseteq$ 2C-f.a., 1'C-f.a. $\subseteq$ 2'C-f.a., 1'C-f.a. $\subseteq$ 3C-f.a., and

7) 2C-f.a. $\subseteq$ 3C-f.a., 2'C-f.a. $\subseteq$ 3C-f.a.

All of the following, which we deduce from the results in sections 4, 4.1, and 5, seem quite hard to show directly:

1) 2C-f.a. are closed under intersection,

2) 1'C-, 2'C-, 3C-n.f.a. are closed under projection,

3) 2C-n.f.a. are not closed under projection,

4) 1C-, 1'C-, 2C-, 2'C-d.f.a. are not closed under complementation (and hence, 1C-n.f.a. $\subset$ 2C-d.f.a., 1C-n.f.a. $\subset$ 2'C-d.f.a., ..., 2'C-n.f.a. $\subset$ 3C-d.f.a.), and

5) 4C-f.a. $\equiv$ 3C-f.a.

There is a well known construction in conventional finite automata theory which, given any nondeterministic automaton on finite strings (n.f.a.f.), determines an equivalent n.f.a.f. with precisely one accepting state. From this same construction it easily follows that corresponding to any 1C-n.f.a. (2C-n.f.a.) there exists an equivalent 1C-n.f.a. (2C-n.f.a., respectively) with precisely one designated state; that corresponding to any 1'C-n.f.a. there exists an equivalent 1'C-n.f.a. with precisely two designated states one of which is the initial state; and that corresponding to any 2'C-n.f.a. there exists an equivalent 2'C-n.f.a. each of whose designated subsets is a singleton set.

I suspect but have not proven:

1) there is a set which is 1C-d.f.a., 1'C-d.f.a., and 2C-d.f.a. definable which is not defined by any 1C-d.f.a., 1'C-d.f.a., or 2C-d.f.a. with only one designated state,

2) there is a set which is both 2'C-d.f.a. and 3C-d.f.a. definable which is not defined by any 2'-d.f.a. or 3-d.f.a. all of whose designated subsets are singleton sets,

3) there is a set which is both 2'-d.f.a. and 3-d.f.a. definable, which is not defined by any 2'-n.f.a. or 3-n.f.a. with only one designated subset.

SECTION X ω-REGULARITY

The following remark is rather obvious and we state it without proof.

Remark 5: $A \subseteq \Sigma^{\omega}$ is 1-f.a. definable iff for some regular event $\alpha \subseteq \Sigma^{*}$, $A = \alpha \cdot \Sigma^{\omega}$.

We have characterized 1'-f.a., 2-d.f.a., and 2'-f.a. definable sets in similar ways. Because these characterizations are much more complicated and seem to be of no real value, we won't present them here.

The set of 2-n.f.a. (3-n.f.a., 3-d.f.a., etc.) definable sets is elegantly characterized as the set of ω-regular events as we now show.

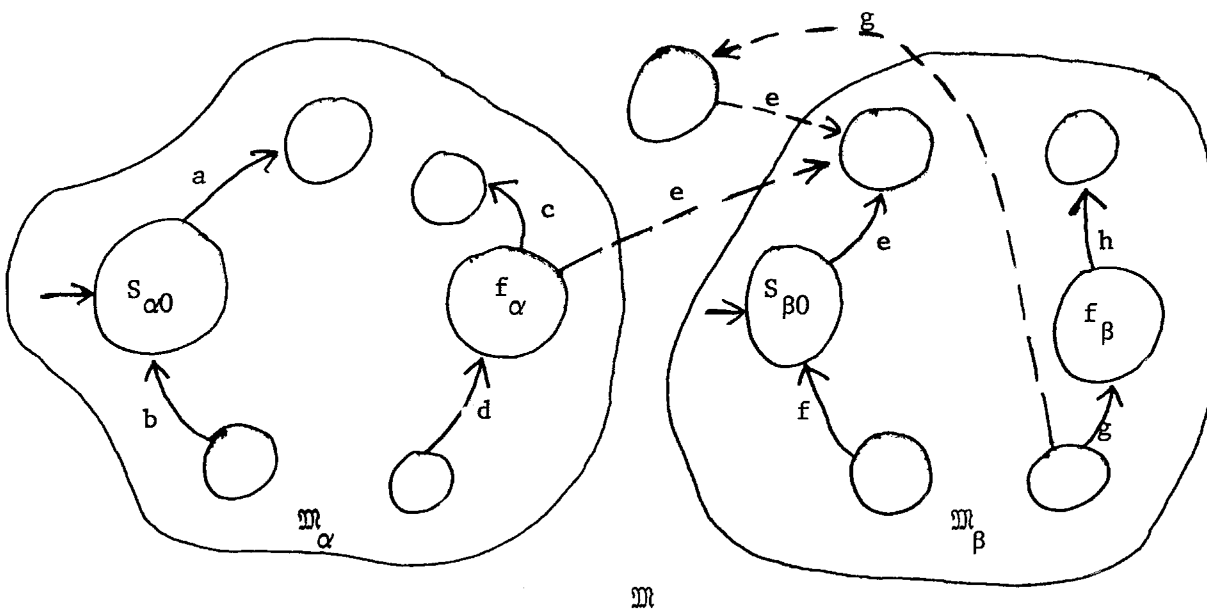
Lemma 16: If $A \subseteq \Sigma^{\omega}$ is an ω-regular event, then A is 2-n.f.a. definable.

Proof: By Theorem 12 the family of 2-n.f.a. definable sets is closed under union. Hence, it suffices to show that the ω-regular event $\alpha\beta^{\omega}$ is 2-n.f.a. definable.

By the definition of ω-regular, $\alpha \subseteq \Sigma^*$ and $\beta \subseteq \Sigma^+$ are regular events. Hence, there exist n.f.a.f.'s $\mathfrak{M}_\alpha = < S_\alpha, \Sigma, M_\alpha, s_{\alpha 0}, \{f_\alpha\} >$ and $\mathfrak{M}_\beta = < S_\beta, \Sigma, M_\beta, s_{\beta 0}, \{f_\beta\} >$ such that $T(\mathfrak{M}_\alpha) = \alpha$ and $T(\mathfrak{M}_\beta) = \beta$.

Using $\mathfrak{M}_\alpha$ and $\mathfrak{M}_\beta$ we construct a 2-n.f.a. defining $\alpha\beta^\omega$ as follows. Define 2-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_{\alpha 0}, \{s_p\} >$, where $S = S_\alpha \cup S_\beta \cup \{s_p\}$, for all $s \in S_\alpha - \{f_\alpha\}$, and all $\sigma \in \Sigma$, $M(s, \sigma) = M_\alpha(s, \sigma)$, $M(f_\alpha, \sigma) = M_\alpha(f_\alpha, \sigma) \cup M_\beta(s_{\beta 0}, \sigma)$, for all $s \in S_\beta$, and all $\sigma \in \Sigma$, $M(s, \sigma) \supseteq M_\beta(s, \sigma)$ and if $f_\beta \in M_\beta(s, \sigma)$ then $s_p \in M(s, \sigma)$; and for all $\sigma \in \Sigma$, $M(s_p, \sigma) = M(s_{\beta 0}, \sigma)$.

For example, a partial state transition diagram of an $\mathfrak{M}$ obtained by the above construction is shown in Figure 4.



Transitions added to $\mathfrak{M}_\alpha$ and $\mathfrak{M}_\beta$ in constructing $\mathfrak{M}$ appear as dashed lines.

FIGURE 4

Suppose $v \in \alpha\beta^{\omega}$. Then there exist $t_0 < t_1 < t_2 \ldots$, such that $w = (v(0) \ldots v(t_0-1)) \in \alpha$, and for all $i \in \mathbb{N}$, $y_i = (v(t_i)\ v(t_i+1) \ldots v(t_{i+1}-1) \in \beta$. Hence, there exists $r \in Rn(\mathfrak{M}_{\alpha}, w)$ such that $r(t_0) = f_{\alpha}$, and for each $i \in \mathbb{N}$, there exists $r_i \in Rn(\mathfrak{M}_{\beta}, y_i)$ such that $r_i(t_{i+1}-t_i) = f_{\beta}$. Hence, from the definition of M, there exists $r \in Rn(\mathfrak{M}, v)$ such that $r(t_0) = f_{\alpha}$, and for all $i > 0$, $r(t_i) = s_p$. Therefore, $s_p \in In(r)$ and r is an accepting $\mathfrak{M}$-run on v. Hence, $v \in T(\mathfrak{M})$.

Suppose $v \in T(\mathfrak{M})$. Then there exists $r \in Rn(\mathfrak{M}, v)$ such that $s_p \in In(r)$. Hence, there exist $t_1 < t_2 < t_3 < \ldots$, such that for all $i > 0$, we have $r(t_i) = s_p$. From the definition of M there must be a $t_0 < t_1$ such that $r(t_0) = f_{\alpha}$. Hence, we have:

$(v(0) \ldots v(t_0-1)) \in \alpha$, and for all $i \in \mathbb{N}$,

$(v(t_i) \ldots v(t_{i+1}-1)) \in \beta$. Therefore, $v \in \alpha\beta^{\omega}$. $\square$

The above proof with trivial modifications suffices to show that 3-n.f.a. $\supseteq$ ω-regular. In fact, a 3-n.f.a. defining $\alpha\beta^{\omega}$ (as above) is $\mathfrak{M} = < S, \Sigma, M, s_{\alpha 0}, \mathcal{F} >$, where S, M, $s_{\alpha 0}$ are as above and $\mathcal{F} = \{F \subseteq S \mid s_p \in F\}$.

<u>Lemma 17</u>: If $A \subseteq \Sigma^{\omega}$ is 2-n.f.a. definable, then A is an ω-regular event.

<u>Proof</u>: Given 2-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, \{s_1, \ldots, s_k\} >$. For each $1 \leq i \leq k$, define 2-n.f.a. $\mathfrak{M}_i = < S, \Sigma, M, s_0, \{s_i\} >$. Clearly, $T(\mathfrak{M}) = T(\mathfrak{M}_1) \cup \ldots \cup T(\mathfrak{M}_k)$. Since ω-regular events are closed under union, the following suffices.

Given 2-n.f.a. $\mathfrak{M} = < S, \Sigma, M, s_0, \{s_1\} >$. Define n.f.a.f.'s $\mathfrak{M}_\alpha = < S, \Sigma, M, s_0, \{s_1\} >$, and $\mathfrak{M}_\beta = < S, \Sigma, M, s_1, \{s_1\} >$. Let $T(\mathfrak{M}_\alpha) = \alpha$ and $T(\mathfrak{M}_\beta) - \{\lambda\} = \beta$. Clearly, it will suffice to show that $T(\mathfrak{M}) = \alpha\beta^\omega$.

Suppose $v \in T(\mathfrak{M})$. Then there exists $r \in Rn(\mathfrak{M}, v)$ such that $s_1 \in In(r)$. Hence, there exist $t_0 < t_1 < t_2 < \ldots$, such that for all $i \in \mathbb{N}$, $r(t_i) = s_1$. Let $w = v(0)\ v(1) \ldots v(t_0-1)$, and for all $i \in \mathbb{N}$, let $y_i = v(t_i)\cdot v(t_i+1) \ldots v(t_{i+1}-1)$. Hence, $v = w\cdot y_0 \cdot y_1 \ldots y_n \ldots$. Let $r_\alpha = r(0)\cdot r(1) \ldots r(t_0)$, and for all $i \in \mathbb{N}$, let $r_i = r(t_i)\ r(t_i+1) \ldots r(t_{i+1})$. We have the picture:

w: $v(0)$ $v(1) \ldots v(t_0-1)$

r_α: $r(0) = s_0$ $r(1) \ldots\ldots\ldots$ $r(t_0) = s_1$

y_0: $v(t_0) \ldots v(t_1-1)$

r_0: $r(t_0) = s_1 \ldots\ldots\ldots\ldots$ $r(t_1) = s_1$

y_i: $v(t_i) \ldots\ldots v(t_{i+1}-1)$

r_i: $r(t_i) = s_1 \ldots\ldots\ldots\ldots$ $r(t_{i+1}) = s_1$

y_{i+1}: $v(t_{i+1}) \ldots$

r_{i+1}: $r(t_{i+1}) = s_1$

Clearly, r_α is an accepting $\mathfrak{M}_\alpha$-run on w, and for all $i \in \mathbb{N}$, r_i is an accepting $\mathfrak{M}_\beta$-run on y_i. Hence, $w \in \alpha$, and for all $i \in \mathbb{N}$, $y_i \in \beta$. Therefore, $v \in \alpha\beta^\omega$.

Suppose $v \in \alpha\beta^\omega$. Then there exist $t_0 < t_1 < t_2 < \ldots$, such that $v(0)\cdot v(1) \ldots v(t_0-1) \in \alpha$, and for all $i \in \mathbb{N}$, $v(t_i)\cdot v(t_i+1) \ldots v(t_{i+1}-1) \in \beta$. Let $w = v(0) \ldots v(t_0-1)$, and for all $i \in \mathbb{N}$, $y_i = v(t_i) \ldots v(t_{i+1}-1)$. There exists r_α an accepting $\mathfrak{M}_\alpha$-run on w, and for all $i \in \mathbb{N}$, there exist r_i an accepting $\mathfrak{M}_\beta$-run on y_i. From the definitions of $\mathfrak{M}_\alpha$ and $\mathfrak{M}_\beta$, we have $r_\alpha(0) = s_0$, $r_\alpha(t_0) = s_1$, and for all $i \in \mathbb{N}$, $r_i(0) = s_1$ and $r_i(t_{i+1}-t_i) = s_1$. We have the picture:

w: $v(0)$ $v(1) \ldots v(t_0-1)$

r_α: $r_\alpha(0) = s_0$ $r_\alpha(1)$ $r_\alpha(t_0) = s_1$

y_0: $v(t_0) \ldots v(t_0-1)$

r_0: $r_0(0) = s_1 \ldots\ldots\ldots r_0(t_1-t_0) = s_1$

y_{i-1}: $\ldots v(t_i-1)$

r_{i-1}: $\ldots r_{i-1}(t_i-t_{i-1}) = s_1$

y_i: $v(t_i) \ldots v(t_{i+1}-1)$

r_i: $r_i(0) = s_1 \ldots\ldots\ldots\ldots r_i(t_{i+1}-t_i) = s_1$

Hence, there exists $r \in Rn(\mathfrak{M}, v)$ such that $r = r_{\alpha} \cdot r_0 \cdot r_1 \ldots r_n \ldots$. Clearly, for all $i \in \mathbb{N}$, $r(t_i) = s_1$. Hence, $s_1 \in In(r)$, r is an accepting $\mathfrak{M}$-run on v, and $v \in T(\mathfrak{M})$.

Therefore, $T(\mathfrak{M}) = \alpha\beta^{\omega}$. □

Theorem 23: $A \subseteq \Sigma^{\omega}$ is 2-n.f.a. definable iff A is an ω-regular event.

Proof: Immediate from Lemmas 16 and 17. □

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Appendix I

The Emptiness Problem for Finite Automata on Infinite Trees

I appreciate Charles Rackoff's extensive help and advice in achieving these results and in writing this presentation of them.

Definition: An f.a.t. (finite automaton on infinite trees) with subset pairs is a system $\mathfrak{A} = < S, \Sigma, M, s_0, \Omega >$, where $< S, \Sigma, M, s_0 >$ is an n-S-table (as defined in Chapter I), and $\Omega = ((R_i, G_i))_{i \le n}$ are the subset pairs.

An $\mathfrak{A}$-run on Σ-tree $t = (v, T_x)$ is any mapping $r: T_x \to S$ such that: 1) $r(x) = s_0$, 2) for all $y \in T_x$, $(r(y0), r(y1)) \in M(r(y), v(y))$.

The set of all $\mathfrak{A}$-runs on t will be denoted by $Rn(\mathfrak{A}, t)$.

An accepting $\mathfrak{A}$-run on $t = (v, T_x)$ is any $r \in Rn(\mathfrak{A}, t)$ such that for all paths $\pi \subset T_x$, $(r \mid \pi)$ is 4-accepting with respect to Ω.

$\mathfrak{A}$ accepts t if there exists an accepting $\mathfrak{A}$-run on t.

$T(\mathfrak{A}) = \{t = (v, T_x) \mid \mathfrak{A} \text{ accepts } t\}$.

An $\mathfrak{A}$-run on finite Σ-tree $e = (v, E_x)$ is any $\mathfrak{A}'$-run on e (as defined in Chapter I), where $\mathfrak{A}' = < S, \Sigma, M, s_0 >$. We denote the set of all $\mathfrak{A}$-runs on e by $Rn(\mathfrak{A}, e)$.

Given f.a.t. with subset pairs $\mathfrak{A} = < S, \Sigma, M, s_0, \Omega >$, we wish to determine whether or not $T(\mathfrak{A}) = \phi$. Consider the f.a.t. with pairs $\bar{\mathfrak{A}} = < S, \{0\}, \bar{M}, s_0, \Omega >$, where for all $s \in S$, $\bar{M}(s, 0) = \bigcup_{\sigma \in \Sigma} M(s, \sigma)$. Clearly, $T(\mathfrak{A}) = \phi$ iff $T(\bar{\mathfrak{A}}) = \phi$.

Thus the emptiness problem is reduced to the case of automata with input alphabet {0}, and henceforth, we restrict our attention to this case. Since for every infinite tree T_x and every finite tree E_x there exists just one {0}-tree $(\bar{v}, T_x)$ and just one finite {0}-tree $(\bar{v}, E_x)$, we will omit mention of the valuation $\bar{v}$ and talk abour $\mathfrak{A}$-runs on T_x and E_x.

Definition: If for some path π we have $x \in \pi$ and $y \in \pi$, then we denote by $[x,y]$ the set $\{w \mid x \le w \le y\}$. Note that when $y < x$, we have $[x,y] = \emptyset$.

Definition: Let α be a string. Let n and m be positive integers such that $n \le m$. Then $\alpha(n)$ denotes the nth element (from the left) of α, and $\alpha([n,m]) = \{\alpha(i) \mid n \le i \le m\}$.

Definition: Let E be a tree (finite or infinite) with root Λ. For $x \in E$, $x = (\sigma_1 \sigma_2 \dots \sigma_k) \in \{0, 1\}^*$, and $r: E \to S$, we denote the S^*-sequence $r(\Lambda) \cdot r(\sigma_1) \cdot r(\sigma_1 \sigma_2) \dots r(x)$ by $\alpha_{r,x}$.

Definition: Let $\mathfrak{A} = < S, \{0\}, M, s_0, ((R_i, G_i))_{i \le n} >$, and let $\alpha \in S^*$ have length p. We say that α is good with respect to $\mathfrak{A}$ if there exist integers h and j such that 1) $h \le j < p$, 2) $\alpha(j) = \alpha(p)$, 3) $\alpha([h,j]) = \alpha([j,p])$, and 4) there exists $i \le n$ such that $\alpha([j,p]) \cap R_i = \emptyset$ and $\alpha(p) \in G_i$.

Theorem 1: For any f.a.t. with subset pairs $\mathfrak{A} = < S, \{0\}, M, s_0, ((R_i, G_i))_{i \le n} >$, $T(\mathfrak{A}) \ne \emptyset \Leftrightarrow$ **for some** finite tree E there exists a mapping $r: E \to S$ such that

1) $r \in Rn(\mathfrak{A}, E)$,

2) there exist mappings $J: Ft(E) \to E\text{-}Ft(E)$ and $H: Ft(E) \to E\text{-}Ft(E)$, such that for all $x \in Ft(E)$,

   a) $H(x) \le J(x) < x$,

   b) $r(J(x)) = r(x)$,

   c) $r([H(x), J(x)]) = r([J(x), x])$, and

   d) for some $i \le n$, $r([J(x), x]) \cap R_i = \phi$ and $r(x) \in G_i$.

Proof: Suppose $T(\mathfrak{A}) \ne \phi$. Then there exists r, an accepting $\mathfrak{A}$-run on T. Clearly, for every path $\pi \subset T$ there exists $x \in \pi$, such that $\alpha_{r,x}$ is good with respect to $\mathfrak{A}$, because for every path $\pi \subset T$ we have $r \mid \pi$ is 4-accepting with respect to $\Omega = ((R_i, G_i))_{i \le n}$. Let $C = \{x \mid \alpha_{r,x}$ is good with respect to $\mathfrak{A}$ and for all $y < x$, $\alpha_{r,y}$ is not good with respect to $\mathfrak{A}\}$. C is a finite frontier. If we let E be the finite tree with frontier C, then clearly from the definition of good string, there exist mappings J and H, which together with $r \mid E$ satisfy conditions 1 and 2 in the statement of Theorem 1. This completes the proof of $\Rightarrow$ in Theorem 1.

$\Leftarrow$

Suppose there exist E, r, J and H satisfying conditions 1 and 2 in the statement of Theorem 1. Then we show that there exists an accepting $\mathfrak{A}$-run on T (and hence, $T(\mathfrak{A}) \ne \phi$) as follows.

Let $\eta: T \to E$ be defined inductively as follows:

1) $\eta(\Lambda) = \Lambda$.

2) if $\eta(x)$ has been defined, then for $\sigma \in \{0, 1\}$,

   a) if $\eta(x) \in E\text{-}Ft(E)$, then $\eta(x\,\sigma) = \eta(x)\cdot\sigma$,

   b) if $\eta(x) \in Ft(E)$, then $\eta(x\,\sigma) = J(\eta(x))\cdot\sigma$.

Let $\bar{r}: T \rightarrow S$ be defined as follows. For all $x \in T$, $\bar{r}(x) = r(\eta(x))$. Clearly, $\bar{r} \in Rn(\mathfrak{A}, T)$ so that it only remains to show that for all paths $\pi \subset T$, $\bar{r} \mid \pi$ is 4-accepting with respect to Ω. That is, $\bar{r}$ is an accepting $\mathfrak{A}$-run on T.

Let $\pi \subset T$ be any specific path. Let $y_0 = \Lambda$, and for all $i < \omega$, let y_{i+1} be the least (under $\leq$) $x \in \pi$ such that $x > y_i$ and $\eta(x) \in Ft(E)$. Clearly, $y_1 y_2 \ldots y_k \ldots$ is the infinite sequence of all nodes in π mapped into $Ft(E)$ by η listed in the order one would encounter them going from the root down along π. Let $v_\pi: \mathbb{N} \rightarrow Ft(E) \times Ft(E)$ be defined as follows. For all $i \in \mathbb{N}$, $v_\pi(i) = (\eta(y_i), \eta(y_{i+1}))$.

For all $i \in \mathbb{N}$ we have by the construction of η:

(I) $\quad J(\eta(y_i)) < \eta(y_{i+1})$, and

(II) $\quad \bar{r}([y_i, y_{i+1}]) = r([J(\eta(y_i)), \eta(y_{i+1})])$. Hence,

(III) $$In(\bar{r} \mid \pi) = \bigcup_{(z_1, z_2) \in In(v_\pi)} r([J(z_1), z_2]).$$

Clearly, there exists a finite sequence $x_0 x_1 x_2 \ldots x_m \in (Ft(E))^*$ such that $x_0 = x_m$ and $In(v_\pi) = \{(x_0, x_1), (x_1, x_2), \ldots, (x_{m-1}, x_m)\}$. Henceforth, let $x_0 x_1 x_2 \ldots x_m$ be a specific such sequence. Let J_i denote $J(x_i)$ and H_i denote $H(x_i)$. By (I), (II), and (III) we have:

(IV) $\quad In(\bar{r} \mid \pi) = \bigcup_{i=0}^{m-1} r([J_i, x_{i+1}])$, and

(V) $\quad$ for all $0 \le i < m$, $J_i < x_{i+1}$.

We now prove three lemmas about finite trees. Then we use the last lemma to complete the proof of Theorem 1.

Lemma 1: There exists M, $0 \le M \le m$, such that for all i, $0 \le i \le m$, we have $H_M \le H_i$. That is, $H_M = \underline{\min}\{H_0, \ldots, H_m\}$.

Induction hypothesis: $(0 \le k < m)$: There exists an integer M', $0 \le M' \le k$, such that for all i, $0 \le i \le k$, $H_{M'} \le H_i$.

Basis: $H_0 \le H_0$.

Induction step:

$H_{M'} \le H_k$, by the induction hypothesis.

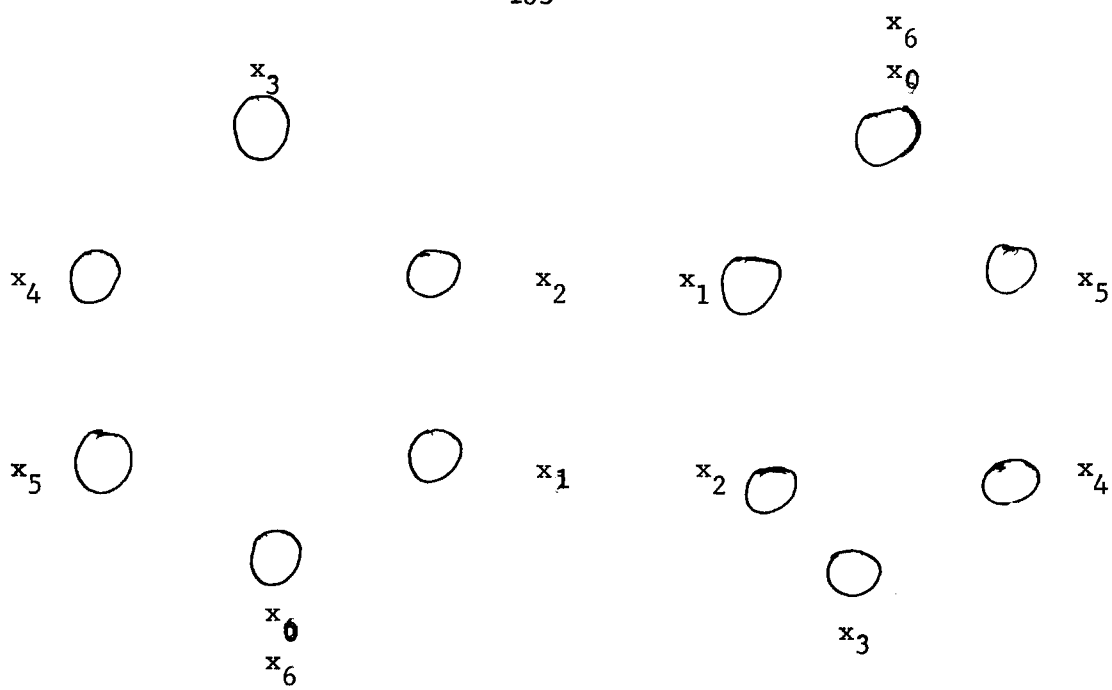
$H_k \le J_k$, by clause 2a in the statement of Theorem 1.

$J_k < x_{k+1}$, by (V).

Hency, $H_{M'} < x_{k+1}$.

By clause 2a in the statement of Theorem 1, we also have $H_{k+1} < x_{k+1}$. Hence, $H_{M'}$ and H_{k+1} are comparable (under $\le$). Clearly, $\underline{\min}\{H_{M'}, H_{k+1}\} \le H_i$, for all i, $0 \le i \le k+1$. □

If $M \neq m$, then we can rename the nodes as indicated in the following diagram for the case M = 3 and m = 6. The nodes are represented as circles O.



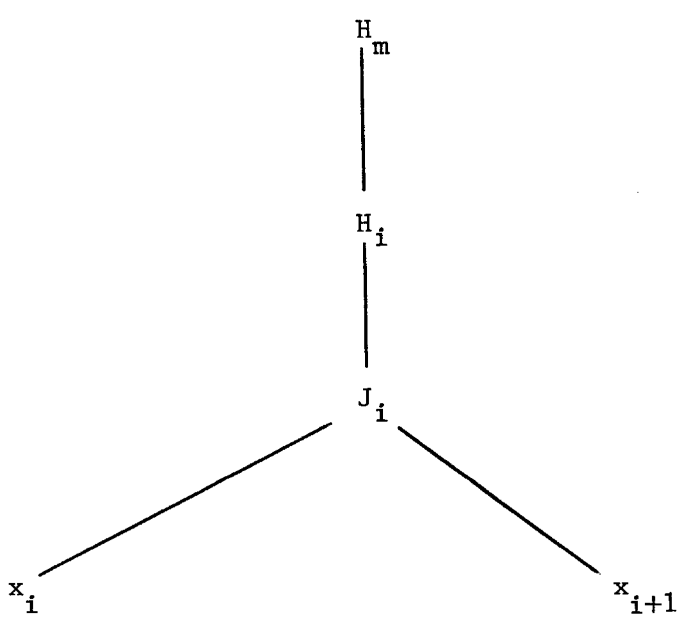
old naming new naming

Clearly, in general after a renaming as above, (IV), (V), and the conditions in the statement of Theorem 1 still hold. Hence, we may assume $H_m = \underline{\min}\{H_0, \ldots, H_m\}$, without loss of generality.

Lemma 2: If $H_m = \underline{\min}\{H_0, \ldots, H_m\}$, then for all i, $0 \le i \le (m-1)$,

$$r([H_m, x_{i+1}]) \supseteq r([H_m, x_i]).$$

Proof: Let i be any integer such that $0 \le i < m$. $H_m \le H_i \le J_i < x_{i+1}$, and $J_i < x_i$, hence we have the picture:

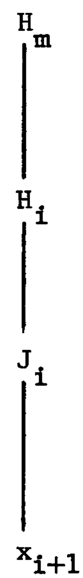


Hence, $r([H_m, x_{i+1}]) \supseteq r([H_m, H_i]) \cup r([H_i, J_i])$. By 2) c) of Theorem 1, $r([H_i, J_i]) = r([J_i, x_i])$. Hence, $r([H_m, x_{i+1}]) \supseteq r([J_i, x_i])$, and therefore, $r([H_m, x_{i+1}]) \supseteq r([H_m, x_i])$. □

Lemma 3: If $H_m = \underline{\min}\{H_0, \ldots, H_m\}$, then for all i, $0 \le i \le (m-1)$, $r([H_m, x_m]) \supseteq r([J_i, x_{i+1}])$.

Proof: Let i be any integer such that $0 \le i \le (m-1)$.

By Lemma 2 $r([H_m, x_m]) \supseteq r([H_m, x_{m-1}])$, $r([H_m, x_{m-1}]) \supseteq r([H_m, x_{m-2}])$, ..., $r([H_m, x_{i+2}]) \supseteq r([H_m, x_{i+1}])$. Hence, $r([H_m, x_m]) \supseteq r([H_m, x_{i+1}])$. We have $H_m \le H_i \le J_i < x_{i+1}$. That is the picture:



Hence, $[H_m, x_{i+1}] \supseteq [J_i, x_{i+1}]$. Hence $r([H_m, x_m]) \supseteq r([J_i, x_{i+1}])$. □

Completion of the Proof of Theorem 1: Without loss of generality we assume $H_m = \underline{\min}\{H_0, \ldots, H_m\}$. By Lemma 3,

$$r([H_m, x_m]) \supseteq \bigcup_{i=0}^{m-1} r([J_i, x_{i+1}]).$$

By part 2) d) of Theorem 1 we have for some i, $0 \le i \le n$, $r([J_m, x_m]) \cap L_i = \emptyset$ and $r(x_m) \in U_i$. By part 2) c) of Theorem 1, $r([H_m, x_m]) = r([J_m, x_m])$. Hence,

$$\bigcup_{i=0}^{m-1} r([J_i, x_{i+1}]) \cap R_i = \emptyset,$$

and

$$\bigcup_{i=0}^{m-1} r([J_i, x_{i+1}]) \cap G_i \neq \emptyset.$$

Therefore, by (II) $(\bar{r} \mid \pi) \in [\Omega]$. □

<u>Theorem 2</u>: The emptiness problem for f.a.t.'s with subset pairs is **decidable.**

<u>Proof</u>: Let $\mathfrak{A} = < S, \{0\}, M, s_0, \Omega >$, be an f.a.t. with subset pairs. Let $B = \{(v, E) \in Y_{\{0\}} \mid$ there exists an $\mathfrak{A}$-run r on (v, E) such that for all $x \in Ft(E)$, $\alpha_{r,x}$ is good with respect to $\mathfrak{A}\}$.

Clearly, we can construct a deterministic finite automaton on finite strings which defines the set of all good strings with respect to $\mathfrak{A}$. Hence, we can determine a 3-f.a.f.t. $\mathfrak{A}_1$ such that $T(\mathfrak{A}_1) = B$. By Theorem 11 of Chapter I, $T(\mathfrak{A}_1) = \emptyset$ is decidable. By Theorem 1 in this appendix, $T(\mathfrak{A}) = \emptyset$ iff $T(\mathfrak{A}_1) = \emptyset$. □

Appendix II

Definition: An f.a.t. with designated subsets is a system $\mathfrak{A} = < S, \Sigma, M, s_0, \mathcal{F} >$, where $< S, \Sigma, M, s_0 >$ is an n-S-table, and $\mathcal{F} \subseteq P(S)$ is the set of designated subsets.

An $\mathfrak{A}$-run on Σ-tree t, Rn($\mathfrak{A}$, t), and $\mathfrak{A}$-run on finite Σ-tree e, and Rn($\mathfrak{A}$, e) are all as in Appendix I for the f.a.t. with subset pairs.

An accepting $\mathfrak{A}$-run on Σ-tree t $= (v, T_x)$ is any $r \in Rn(\mathfrak{A}, t)$ such that for all paths $\pi \subset T_x$, $In(r \mid \pi) \in \mathcal{F}$.

$\mathfrak{A}$ accepts t if there is an accepting $\mathfrak{A}$-run on t.

$T(\mathfrak{A}) = \{t = (v, T_x) \mid$ there exists an accepting $\mathfrak{A}$-run on t$\}$.

Theorem 3: For any f.a.t. with designated subsets $\mathfrak{A} = < S, \{0\}, M, s_0, \mathcal{F} >$, $T(\mathfrak{A}) \neq \emptyset \Leftrightarrow$ for some finite tree E there exists a mapping $r: E \to S$ such that

1) $r \in Rn(\mathfrak{A}, E)$,

2) there exist mappings $J: Ft(E) \to E\text{-}Ft(E)$ and $H: Ft(E) \to E\text{-}Ft(E)$ such that for all $x \in Ft(E)$

 a) $H(x) \leq J(x) < x$,

 b) $r(J(x)) = r(x)$, and

 c) $r([H(x), J(x)]) = r([J(x), x]) \in \mathcal{F}$.

Proof: The proof of the implication to the right ($\Rightarrow$) is essentially the same as in Theorem 1 of Appendix I.

The proof of the implication to the left ($\Leftarrow$) follows from a simple extension of the proof of Theorem 1 using the following lemma.

The following lemma is independent of Lemmas 1, 2, and 3 of Appendix I. In extending the proof of Theorem 1 to a proof of Theorem 2, Lemma 4 could be reasonably placed just before Lemma 1 or just after Lemma 3. Lemma 4 should be evaluated as if it appeared in the just mentioned context. In the following, let $x_0, \ldots, x_m$ be as in Lemmas 1, 2, and 3 in Appendix I.

Lemma 4: For all k, $0 \le k \le m$, $\bigcup_{i=0}^{m-1} [J_i, x_{i+1}] \supseteq [J_k, x_k]$.

(Note that Lemma 4 is another lemma about finite trees, as were Lemmas 1, 2, and 3. Note, also, that unlike Lemmas 2 and 3, Lemma 4 does not involve the $\mathfrak{A}$-run r. Lemma 4 is simply about sets of nodes of T.)

Proof: The facts about the sequence of nodes $x_0 x_1 \ldots x_m$ which the following proof uses are:

(I) $$x_0 = x_m, \text{ and for all } i,\ 0 \le i < m,\ J_i < x_i \ \&\ J_i < x_{i+1}.$$

Definition: For $0 \le i, j \le m$, let the branching point B(i, j) be the g.l.b. (under $\le$) in T of $\{x_i, x_j\}$. That is, B(i, j) is the unique $y \in T$ such that 1) $y \le x_i \ \&\ y \le x_j$, and 2) $(\forall w \in T)(w \le x_i \ \&\ w \le x_j \rightarrow w \le y)$.

Observe that for all $0 \le i < m$, $J_i < x_i \ \&\ J_i < x_{i+1}$, and hence, $J_i \le B(i, i+1)$.

As noted in Appendix I, we can rename the nodes, $x_0, x_1, \ldots, x_m$, so that node x_k becomes node x_0 and (I) holds for the new sequence $x_0 x_1 \ldots x_m$. Hence, we prove Lemma 4 by proving:

(II) $\bigcup_{i=0}^{m-1} [J_i, x_{i+1}] \supseteq [J_0, x_0]$.

We prove (II) by proving by induction on h, $0 \le h \le (m-1)$,

(III) $\bigcup_{i=0}^{h} [J_i, x_{i+1}] \supseteq [J_0, B(0, h+1]$;

for when h = m-1 this gives us $\bigcup_{i=0}^{m-1} [J_i, x_{i+1}] \supseteq [J_0, x_0]$ (since $x_m = x_0$ and hence, $B(0, m) = x_0$).

Therefore, the following induction completes the proof of Lemma 4.

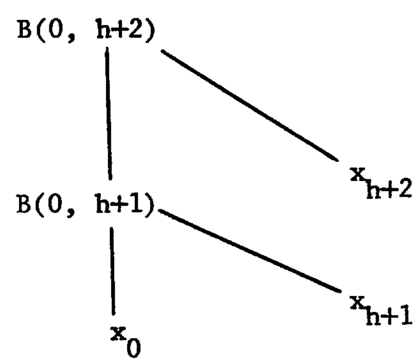
<u>Induction hypothesis</u>: $(0 \le h \le m-1))$: $\bigcup_{i=0}^{h} [J_i, x_{i+1}] \supseteq [J_0, B(0, h+1)]$.

<u>Basis</u>: $[J_0, x_0] \supseteq [J_0, B(0, 1)]$.

<u>Induction step</u>: $B(0, h+2) < x_0$ & $B(0, h+1) < x_0$ hence, B(0, h+1) and B(0, h+2) are comparable, so that either case 1 or case 2 holds.

<u>Case 1</u>: $B(0, h+2) \le B(0, h+1)$.

We have the picture:



From the picture it is clear that $[J_0, B(0, h+1)] \supseteq [J_0, B(0, h+2)]$; and hence by the induction hypothesis,

Clearly, we have $[J_{h+1}, x_{h+2}] \supseteq [B(0, h+1), B(0, h+2)]$. Since $[J_0, B(0, h+2)] \subseteq [J_0, B(0, h+1)] \cup [B(0, h+1), B(0, h+2)]$, we have by the induction hypothesis,

$$\bigcup_{i=0}^{h+1} [J_i, x_{i+1}] \supseteq [J_0, B(0, h+2)].$$

This completes the induction and the proof of Lemma 2. □

The extension of the proof of Theorem 1 needed to prove Theorem 3 is as follows.

By Lemmas 3 and 4, if $H_m = \underline{\min}\{H_0, \ldots, H_m\}$, then

$$\bigcup_{i=0}^{m-1} r([J_i, x_{i+1}]) = r([J_m, x_m]).$$

Then by clause 2c in the statement of Theorem 3,

$$\mathrm{In}(\bar{r} \mid \pi) = \bigcup_{i=0}^{m-1} r([J_i, x_{i+1}]) = r([J_m, x_m]) \in \mathcal{F}.$$ This completes the proof of Theorem 3. □

<u>Theorem 4</u>: The emptiness problem for f.a.t.'s with designated subsets is decidable.

The proof of Theorem 4 using an appropriate definition of good string and Theorem 3 is essentially the same as the proof of Theorem 2 in Appendix I.

<u>Remark</u>: Let f.a.t. $\mathfrak{A}$ (with subset pairs or with designated subsets have q states. For either definition of good string (i.e., the definition appropriate for the proof of Theorem 2 or the definition appropriate to Theorem 4) we can construct a nondeterministic finite automaton on finite strings, $\mathfrak{M}$, which defines the set of good strings and which has at most $2^{2q}(q+1)$ states. By the subset construction we can design a deterministic finite automaton $\mathfrak{M}_1$ equivalent to $\mathfrak{M}$ and such that $\mathfrak{M}_1$ has at most $2^{2^{2q}(q+1)}$ states. Using $\mathfrak{M}_1$ we can easily construct a 3-f.a.f.t. $\mathfrak{A}_1$ such that $T(\mathfrak{A}_1) = \phi$ iff $T(\mathfrak{A}) = \phi$, and such that the state set of $\mathfrak{A}_1$ is the cross product of the state sets of $\mathfrak{A}$ and $\mathfrak{M}_1$. Hence, $\mathfrak{A}_1$ has at most Q states where $Q = q\ 2^{2^{2q}(q+1)}$. By Theorem 11 of Chapter I, we can determine whether $T(\mathfrak{A}_1) = \phi$ in Q^3 computational steps.

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13. ABSTRACT

Chapter I is a survey of finite automata as acceptors of finite labeled trees. Chapter II is a survey of finite automata as acceptors of infinite strings on a finite alphabet. Among the automata models considered in Chapter II are those used by McNaughton, Buchi, and Landweber. In Chapter II we also consider several new automata models based on a notion of a run of a finite automaton on an infinite string suggested by Professor A. R. Meyer in private communication. We show that these new models are all equivalent to various previously formulated models.

M. O. Rabin has published two solutions of the emptiness problem for finite automata operating on infinite labeled trees. Appendices I and II contail a new solution of this emptiness problem. This new solution was obtained jointly by the author and Charles Rackoff.

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